MATHEMATICS

STUDENT'S TEXTBOOK
GRADE 9

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FEDERAL DEMOCRATIC REPUBLIC OF ETHIOPIA
MINISTRY OF EDUCATION

MATHEMATICS STUDENT' S TEXTBOOK GRADE 9

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MATHEMATICS

STUDENT TEXTBOOK GRADE 9

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Welcoming Message to Students.

Dear grade 9 students, you are welcome to the first grade of secondary level education. This is a golden stage in your academic career. Joining secondary school is a new experience and transition from primary school Mathematics education. In this stage, you are going to get new knowledge and experiences which can help you learn and advance your academic, personal, and social career in the field of Mathematics.

Enjoy it!

Introduction on Students' Textbook.

Dear students, this textbook has 9 units namely: Further on sets, the number system, Solving Equations, Solving Inequalities, Introduction to Trigonometry, Regular Polygons, Congruency and Similarity, Vectors in two Dimensions and Statistics and Probability respectively. Each of the units is composed of introduction, objectives, lessons, key terms, summary, and review exercises. Each unit is basically unitized, and lesson based. Structurally, each lesson has four components: Activity, Definition, Examples, and Exercises (ADEE).

The most important part in this process is to practice problems by yourself based on what your teacher shows and explains. Your teacher will also give you feedback, assistance and facilitate further learning. In such a way you will be able to not only acquire new knowledge and skills but also advance them further. Basically, the four steps of each of the lessons are: Activity, Definition/Theorem/Note, Example and Exercises.

Activity

This part of the lesson demands you to revise what you have learnt or activate your background knowledge on the topic. The activity also introduces you what you are going to learn in new lesson topic.

Definition/Theorem/Note

This part presents and explains new concepts to you. However, every lesson may not begin with definition, especially when the lesson is a continuation of the previous one.

Example and Solution

Here, your teacher will give you specific examples to improve your understanding of the new content. In this part, you need to listen to your teacher's explanation carefully and participate actively. Note that your teacher may not discuss all the examples in the class. In this case, you need to attempt and internalize the examples by yourself.

Exercise

Under this part of the material, you will solve the exercise and questions individually, in pairs or groups to practice what you learnt in the examples. When you are doing the exercise in the classroom either in pairs or groups, you are expected to share your opinions with your friends, listen to others' ideas carefully and compare yours with others. Note that you will have the opportunity of cross checking your answers to the questions given in the class with the answers of your teacher. However, for the exercises not covered in the class, you will be given as a homework, assignment, or project. In this case, you are expected to communicate your teacher for the solutions.

This symbol indicates that you need some time to remember what you have learnt before or used to enclose steps that you may be encouraged to perform mentally. This can help you connect your previous lessons with what it will come in the next discussions.

Contents

UNIT 1

FURTHER ON SETS

Unit Outcomes

By the end of this unit, you will be able to

- **Explain facts about sets.**
- **Describe sets in different ways.**
- **Define operations on sets.**
- **Demonstrate set operations using Venn diagram.**
- **Apply rules and principles of set theory for practical situations.**

Unit Contents

1.1 **Sets and Elements**

1.2 **Set Description**

1.3 **The Notion of Sets**

1.4 **Operations on Sets**

1.5 **Application**

Summary

Review Exercise

KEY Words

- subset
- symmetric difference
- set description
- Venn diagram
- empty set
- intersection
- absolute complement
- union
- proper subset
- complement set

Introduction

In Grade 7 you have learnt basic definition and operations involving sets. The concept of a set serves as a fundamental part of the present day mathematics. Today this concept is being used in almost every branch of mathematics. We use sets to define the concepts of relations and functions.

In this unit, you will discuss some further definitions, operations and applications involving sets.

Activity 1.1

1. Define a set in your own words.
2. Which of the following are well defined sets and which are not? Justify your answer.
   a. Collection of students in your class.
   b. Collection of beautiful girls in your class.
   c. Collection of consonants of the English alphabet.
   d. Collection of hardworking teachers in a school.

1.1 Sets and Elements

A set is a collection of well-defined objects or elements. When we say a set is well-defined, we mean that if an object is given, we are able to determine whether the object is in the set or not.

Note

i) Sets are usually denoted by capital letters like A, B, C, X, Y, Z, etc.

ii) The elements of a set are represented by small letters like a, b, c, x, y, z, etc.

If a is an element of set A, we say "a belongs to A". The Greek symbol ∈ (epsilon) is used to denote the phrase "belongs to". Thus, we write $\boldsymbol{a} \in \boldsymbol{A}$ if a is a member of set A. If b is not an element of set A, we write $\boldsymbol{b} \notin \boldsymbol{A}$ and read as "b does not belong to set A" or "b is not a member of set A".

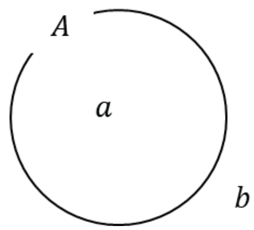


Figure 1.1

Example 1

a. The set of students in your class is a well-defined set since the elements of the set are clearly known.

b. The collection of kind students in your school. This is not a well-defined set because it is difficult to list members of the set.

c. Consider G as a set of vowel letters in English alphabet. Then $a \in G, o \in G, i \in G$, but $b \notin G$.

Example 2

Suppose that A is the set of positive even numbers. Write the symbol ∈ or ∉ in the blank spaces.

a. 4 _____ A **b.** 5_____ A **c.** −2_____ A **d.** 0____ A

Solution:

The positive even numbers include 2, 4, 6, 8, ... Therefore,

a. $4 \in A$ **b.** $5 \notin A$, **c.** $-2 \notin A$, **d.** $0 \notin A$.

Exercise 1.1

1. Which of the following is a well-defined set? Justify your answer.
   a. A collection of all boys in your class.
   b. A collection of efficient doctors in Black Lion Hospital.
   c. A collection of all natural numbers less than 100.
   d. The collection of songs by Artist Tilahun Gessese.
2. Let $A = \{1, 2, 3, 4, 5, 6\}$. Insert the appropriate symbol $\in$ or $\notin$ in the blank spaces:
   a. 5 ___ A b. 8 ___ A c. 0 ___ A d. 4 ___ A e. 7 ___ A

1.2 Set Description

Sets can be described in the following ways.

i) Verbal method (Statement form)

In this method, the well-defined description of the elements of the set is written in an ordinary English language statement form (in words).

Example 1

a. The set of whole numbers greater than 1 and less than 20.

b. The set of students in this mathematics class.

ii) Listing Methods

a) Complete listing method (Roster Method)

In this method, all elements of the set are completely listed. The elements are separated by commas and are enclosed within set braces, { }.

Example 2

a. The set of all even positive integers less than 7 is described in complete listing method as $\{2, 4, 6\}$.

b. The set of all vowel letters in the English alphabet is described in complete listing method as $\{a, e, i, o, u\}$.

b) Partial listing method

We use this method, if listing of all elements of a set is difficult or impossible but the elements can be indicated clearly by listing a few of them that fully describe the set.

Example 3

Use partial listing method to describe the following sets.

a. The set of natural numbers less than 100.

b. The set of whole numbers.

Solution:

a. The set of natural numbers less than 100 are $1, 2, 3, \ldots, 99$. So, naming the set as A, we can express A by partial listing method as $A = \{1, 2, 3, \ldots, 99\}$. The three dots after element 3 and the comma above indicate that the elements in the set continue in that manner up to 99.

b. Naming the set of whole numbers by $\mathbb{W}$, we can describe it as

$\mathbb{W} = \{0, 1, 2, 3, \ldots\}$.

So far, you have learnt three methods of describing a set. However, there are sets which cannot be described by these three methods. Here, below is another method of describing a set.

iii) Set builder method (Method of defining property)

The set-builder method is described by a property that its member must satisfy the common property. This is the method of writing the condition to be satisfied by a set or property of a set.

In set brace, write the representative of the elements of a set, for example x, and then write the condition that x should satisfy after the vertical line (|) or colon (:)

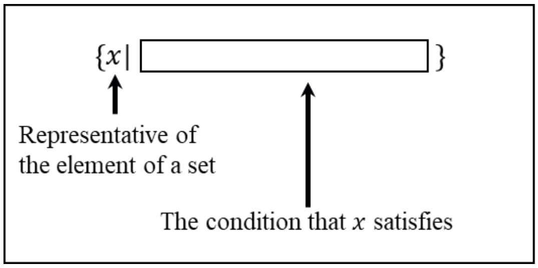


Figure 1.2

Note

The set of natural numbers, whole numbers, and integers are denoted by $\mathbb{N}$, $\mathbb{W}$, and $\mathbb{Z}$, respectively. They are defined as

$$\mathbb{N} = \{1, 2, 3, \ldots\}$$
$$\mathbb{W} = \{0, 1, 2, 3, \ldots\},$$
$$\mathbb{Z} = \{\ldots -3, -2, -1, 0, 1, 2, 3, \ldots\}.$$

Example 4

Describe the following sets using set builder method.

i) Set $A = \{1, 2, 3 \ldots 10\}$ can be described in set builder method as:

$A = \{x \mid x \in \mathbb{N} \text{ and } x < 11\}$. We read this as "$A$ is the set of all elements of natural numbers less than 11."

ii) Let set $B = \{0, 2, 4, \ldots.\}$. This can be described in set builder method as:

$B = \{x \mid x \in \mathbb{Z}$ and x is a non-negative even integer$\}$ or

$B = \{2x \mid x = 0, 1, 2, 3, \ldots\}$ *or* $B = \{2x \mid x \in \mathbb{W}\}$.

Exercise 1.2

1. Describe each of the following sets using a verbal method.
   a. $A = \{5, 6, 7, 8, 9\}$
   b. $M = \{2, 3, 5, 7, 11, 13\}$
   c. $G = \{8, 9, 10, \ldots.\}$
   d. $E = \{1, 3, 5, \ldots, 99\}$
2. Describe each of the following sets using complete and partial listing method (if possible):
   a. The set of positive even natural numbers below or equal to 10.
   b. The set of positive even natural numbers below or equal to 30.
   c. The set of non-negative integers.
   d. The set of even natural numbers.
   e. The set of natural numbers less than 100 and divisible by 5.
   f. The set of integers divisible by 3.
3. List the elements of the following sets:

a. $A = \{3x \mid x \in \mathbb{W}\}$ b. $B = \{x \mid x \in \mathbb{N} \text{ and } 5 < x < 10\}$

4. Write the following sets using set builder method.

a. $A = \{1, 3, 5 \ldots. \}$ b. $B = \{2, 4, 6, 8\}$

c. $C = \{1, 4, 9, 16, 25\}$ d. $D = \{4, 6, 8, 10, \ldots, 52\}$

e. $E = \{-10, \ldots, -3, -2, -1, 0, 1, 2, \ldots, 5\}$ f. $F = \{1, 4, 9, \ldots.\}$

1.3 The Notion of Sets

Empty set, Finite set and Infinite set

Empty Set

A set which does not contain any element is called an **empty set**, **void set** or **null set**. The empty set is denoted mathematically by the symbol { } or Ø.

Example 1

Let set $A = \{x \mid 1 < x < 2, x \in \mathbb{N}\}$. Then, A is an empty set, because there is no natural number between numbers 1 and 2.

Finite set and Infinite set

Definition 1.1

A set which consists of a definite number of elements is called a finite set. A set which is not finite is called an infinite set.

Example 2

Identify the following sets as finite set or infinite set.

a. The set of natural numbers up to 10 **b.** The set of African countries

c. The set of whole numbers

Solution:

a. Let A be a set and $A = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$.

Thus, it is a finite set because it has definite (limited) number of elements.

b. The set of African countries is a finite set.

c. The set of whole numbers is an infinite set.

Note

The number of elements of set A is denoted by $n(A)$. For instance, in the above example 2a, $n(A) = 10$. Read $n(A)$ as number of elements of set A

Exercise 1.3

1. Identify empty set from the list below.
   a. $A = \{x \mid x \in \mathbb{N} \text{ and } 5 < x < 6\}$
   b. $B = \{0\}$
   c. C is the set of odd natural numbers divisible by 2.
   d. $D = \{\,\}$
2. Sort the following sets as finite or infinite sets.
   a. The set of all integers
   b. The set of days in a week
   c. $A = \{x : x \text{ is a multiple of } 5\}$
   d. $B = \{x : x \in Z, x < -1\}$
   e. $D = \{x : x \text{ is a prime number}\}$

Equal Sets, Equivalent Sets, Universal Set, Subset and Proper Subset

Equal Sets

Definition 1.2

Two sets A and B are said to be equal if and only if they have exactly the same or identical elements. Mathematically, it is denoted as $A = B$.

Example 1

Let $A = \{1, 2, 3, 4\}$ and $B = \{4, 3, 2, 1\}$. Then, $A = B$. Set A and set B are equal.

Equivalent Sets

Definition 1.3

Two sets A and B are said to be equivalent if there is a one-to-one correspondence between the two sets. This is written mathematically as $A \leftrightarrow B$ (or $A \sim B$).

Note

Observe that two finite sets A and B are equivalent, if and only if they have equal number of elements and we write mathematically this as $n(A) = n(B)$.

Example 2

Consider two sets $A = \{1, 2, 3, 4\}$ and $B = \{\text{Red}, \text{Blue}, \text{Green}, \text{Black}\}$.

In set A there are four elements and in set B also there are four elements. Therefore, set A and set B are equivalent.

Universal Set (U)

Definition 1.4

A **universal set** (usually denoted by **U**) is a set which has elements of all the related sets, without any repetition of elements.

Example 3

Let set $A = \{2, 4, 6, \ldots\}$ and $B = \{1, 3, 5, \ldots\}$. The universal set U consists of all natural numbers, such that $U = \{1, 2, 3, 4, \ \ldots\}$. Therefore, as we know all even and odd numbers are part of natural numbers. Hence, set U has all the elements of set A and set B.

Subset (⊆)

Definition 1.5

Set A is said to be a subset of set B if every element of A is also an element of B. Figure 1.3 shows this relationship. Mathematically, we write this as $\boldsymbol{A} \subseteq \boldsymbol{B}$. If set A is not a subset of set B, then it is written as $A \nsubseteq B$.

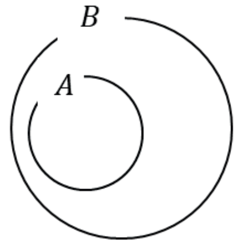


Figure 1.3

Example 4

Let $A = \{1, 2, 3\}$ and $B = \{1, 2, 3, 4\}$ be sets. Here, set A is a subset of set B, or $A \subseteq B$, since all members of set A are found in set B.

In the above set A, find all subsets of the set. How many subsets does set A have?

Solution:

The subsets of set A are $\{1\}, \{2\}, \{3\}, \{1, 2\}, \{2, 3\}, \{1, 3\}, \{1, 2, 3\}, \{\ \}$. The number of subsets of set A is 8.

Note

i) Any set is a subset of itself.

ii) Empty set is a subset of every set.

iii) If set A is finite with n elements, then the number of subsets of set A is 2^n.

In the above Example 3.b, $n(A) = 3$. Then, the number of subsets is $2^3 = 8$.

Proper Subset (⊂)

Definition 1.6

If $A \subseteq B$ and $A \neq B$, then A is called the proper subset of set B and it can be written as $A \subset B$.

Example 5

Given that sets $A = \{2,5,7\}$ and $B = \{2,5,7,8\}$. Set A is a proper subset of set B, that is, $A \subset B$ since $A \subseteq B$ and $A \neq B$. Observe also that $B \not\subset$ A.

In the above set A, find all the proper subsets. How many proper subsets does set A have?

Solution:

The proper subsets of set A are $\{2\}, \{5\}, \{7\}, \{2,5\}, \{5,7\}, \{2,7\}, \{\ \}$. There are seven subsets.

Note

i) For any set A, A is not a proper subset of itself.

ii) The number of proper subsets of set A is $2^n - 1$.

iii) Empty set is the proper subset of any other sets.

iv) If set A is subset of set B $(A \subseteq B)$, conversely B is super set of A written as $B \supset A$.

Exercise 1.4

1. Identify equal sets, equivalent sets or which are neither equal nor equivalent.
   a. $A = \{1,2,3\}$ and $B = \{4,5\}$
   b. $C = \{q,s,m\}$ and $D = \{6,9,12\}$
   c. $E = \{3,7,9,11\}$ and $F = \{3,9,7,11\}$
   d. $G = \{I,J,K,L\}$ and $H = \{J,K,I,L\}$
   e. $I = \{x \mid x \in \mathbb{W},\ x < 5\}$ and $J = \{x \mid x \in \mathbb{N}, x \leq 5\}$
   f. $K = \{x \mid x \text{ is a multiple of } 30\}$ and $L = \{x \mid x \text{ is a factor of } 10\}$
2. List all the subsets of set $H = \{1,3,5\}$. How many subsets and how many proper subsets does it have?
3. Determine whether the following statements are true or false.
   a. $\{a,\ b\} \not\subset \{b,c,a\}$
   b. $\{a,\ e\} \subseteq \{x \mid x \text{ is a vowel in the English alphabet}\}$ c. $\{a\} \subset \{a,b,c\}$

4. Express the relationship of the following sets, using the symbols $\subset$, $\supset$, or $=$

a. $A = \{1, 2, 5, 10\}$ and $B = \{1, 2, 4, 5, 10, 20\}$

b. $C = \{x \mid x$ is natural number less than $10\}$ and $D = \{1, 2, 4, 8\}$

c. $E = \{1, 2\}$ and $F = \{x \mid 0 < x < 3, x \in \mathbb{Z}\}$

5. Consider sets $A = \{2, 4, 6\}$, $B = \{1, 3\ 7, 9, 11\}$ and $C = \{4, 8, 11\}$, then

a. Find the universal set

b. Relate sets A, B, C and U using subset.

1.4 Operations on Sets

There are several ways to create new sets from sets that have already been defined. Such process of forming new set is called set operation. The three most important set operations namely Union$(\cup)$, Intersection$(\cap)$, Complement$(')$ and Difference $(-)$ are discussed below.

Union and Intersection

Activity 1.2

Let the universal set is the set of natural numbers $\mathbb{N}$ which is less than 12, and sets $A = \{1, 2, 3, 4, 5, 6\}$ and $B = \{1, 3, 5, 7, 9\}$.

a. Can you write a set consisting of all natural numbers that are in A or in B?

b. Can you write a set consisting of all natural numbers that are in A and in B?

c. Can you write a set consisting of all natural numbers that are in A and not in B?

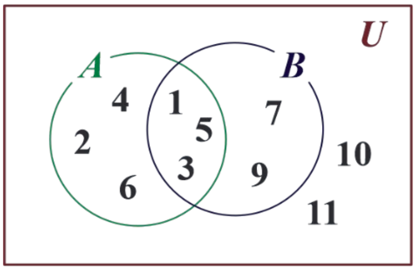


Figure 1.4

Venn diagrams

A Venn diagram is a schematic or pictorial representation of the sets involved in the discussion. Usually sets are represented as interlocking circles, each of which is enclosed in a rectangle, which represents the universal set. Figure 1.4 above is an example of Venn diagram.

Definition 1.7

The union of two sets A and B, which is denoted by $\boldsymbol{A \cup B}$, is the set of all elements that are either in set A or in set B (or in both sets). We write this mathematically as

$A \cup B = \{x \mid x \in A \text{ or } x \in B\}$.

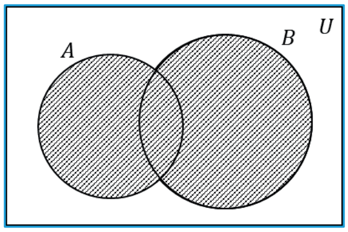


Figure 1.5

Definition 1.8

The intersection of two sets A and B, denoted by $A \cap B$, is the set of all elements that are both in set A and in set B. We write this mathematically as

$$A \cap B = \{x \mid x \in A \text{ and } x \in B\}$$

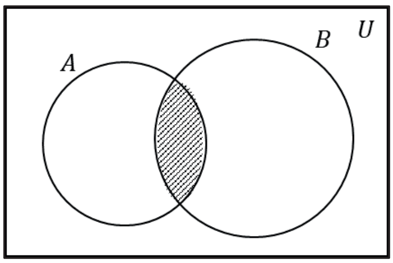


Figure 1.6

Note

Two sets A and B are disjoint if $A \cap B = \emptyset$

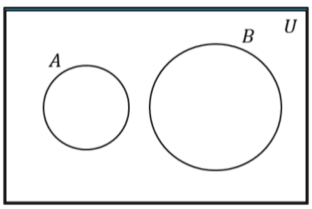


Figure 1.7

Example 1

Let $A = \{0, 1, 3, 5, 7\}$ and $B = \{1, 2, 3, 4, 6, 7\}$ be sets. Draw the Venn diagram and find $A \cup B$ and $A \cap B$.

Solution:

Figure 1.8 shows Venn diagram of set A and B.

Thus, $A \cup B = \{0, 1, 2, 3, 4, 5, 6, 7\}$ and

$A \cap B = \{1, 3, 7\}$.

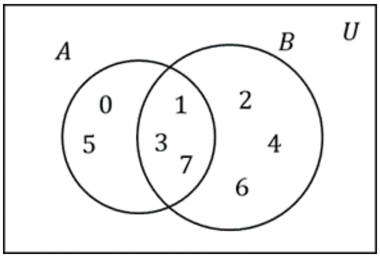


Figure 1.8

Example 2

Let $A = \{2, 4, 6, 8, 10, ...\}$ and $B = \{3, 6, 9, 12, 15, ...\}$ be sets. Then, find $A \cup B$ and $A \cap B$.

Solution:

$A \cup B = \{x \mid x$ is a positive integer that is either even or a multiple of 3$\}$

$= \{2, 3, 6, 9, 12, 15, \ldots.\}$

$A \cap B = \{x \mid x$ is a positive integer that is both even and a multiple of 3$\}$

$= \{6, 12, 18, 24, \ldots.\}$

Note

i) Law of $\emptyset$ and U: $\emptyset \cap A = \emptyset$, $U \cap A = A$.

ii) Commutative law: $A \cap B = B \cap A$.

iii) Associative Law: $(A \cap B) \cap C = A \cap (B \cap C)$.

Exercise 1.5

1. Let $A = \{0, 2, 4, 6, 8\}$ and $B = \{0, 1, 2, 3, 5, 7, 9\}$. Draw the Venn Diagram and find $A \cup B$ and $A \cap B$.
2. Let A be the set of positive odd integers less than 10 and B is the set of positive multiples of 5 less than or equal to 20. Find a) $A \cup B$, b) $A \cap B$.
3. Let $C = \{x \mid x$ is a factors of 20$\}$, $D = \{y \mid y$ is a factor of 12$\}$. Find a) $C \cup D$, b) $C \cap D$.

Complement of sets

Definition 1.9

Let A be a subset of a universal set U. The absolute complement (or simply complement) of A, which is denoted by $\boldsymbol{A'}$, is defined as the set of all elements of U that are not in A. We write this mathematically as

$$A' = \{x: x \in U \text{ and } x \notin A\}.$$

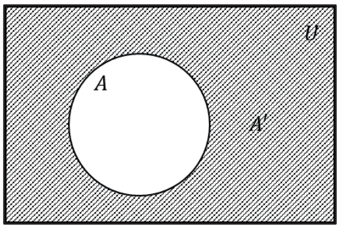


Figure 1.9

Example 1

a. Let $U = \{0, 1, 2, 3, 4\}$ and $A = \{3, 4\}$. Then, $A' = \{0, 1, 2\}$.

Example 2

Let $U = \{1, 2, 3, \dots, 10\}$ be a universal set,

$A = \{x \mid x \text{ is a positive factor of } 10 \text{ in } U\}$ and

$B = \{x \mid x \text{ is an odd integer in } U\}$ be sets.

a. Find A' and B'.

b. Find $(A \cup B)'$ and $A' \cap B'$. What do you observe from the answers?

Solution:

a. $A = \{1, 2, 5, 10\}$, $B = \{1, 3, 5, 7, 9\}$, Thus,

$A' = \{3, 4, 6, 7, 8, 9\}$, $B' = \{2, 4, 6, 8, 10\}$.

b. First, we find $A \cup B$. Hence, $A \cup B = \{1, 2, 3, 5, 7, 9, 10\}$ and

$$(A \cup B)' = \{4, 6, 8\}.$$

On the other hand, from A' and B', we obtain $A' \cap B' = \{4, 6, 8\}$. Hence, we immediately observe $(A \cup B)' = A' \cap B'$.

In general, for any two sets A and B, $(A \cup B)' = A' \cap B'$. It is called the first statement of De Morgan's law.

De Morgan's Law

For the complement set of $A \cup B$ and $A \cap B$,

1st statement: $(A \cup B)' = A' \cap B'$,

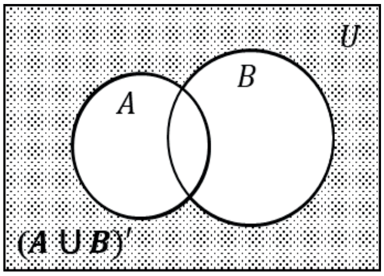


Figure 1.10

2nd statement: $(A \cap B)' = A' \cup B'$.

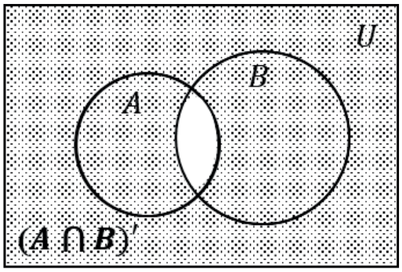


Figure 1.11

Exercise 1.6

1. If the universal set $U = \{0, 1, 2, 3, 4, 5\}$, and $A = \{4, 5\}$, then find A'.
2. Let the universal set $U = \{1, 2, 3, \ldots, 20\}$, $A = \{x \mid x \text{ is a positive factor of } 20\}$ and $B = \{x \mid x \text{ is an odd integer in } U\}$. Find A', B', $(A \cup B)'$ and $A' \cap B'$.
3. Let the universal set be $U = \{x \mid x \in \mathbb{N}, x \leq 10\}$. When $A = \{2, 5, 9\}$, and $B = \{1, 5, 6, 8\}$, find a) $A' \cap B'$ and b) $A' \cup B'$.

Difference of sets

Definition 1.10

The difference between two sets A and B, which is denoted by $A - B$, is the of all elements in A and not in B; this set is also called the relative complement of A with respect to B. We write this mathematically as $A - B = \{x \mid x \in A \text{ and } x \notin B\}$.

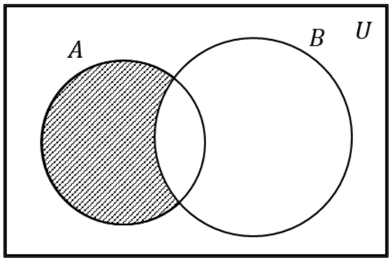


Figure 1.12

Note

The notation $A - B$ can be also written as $A \backslash B$.

Example 1

If sets $A = \{0, 1, 2, 3, 4\}$ and $B = \{3, 4\}$, then $A - B$ or $A \backslash B = \{0, 1, 2\}$.

Example 2

Let U be a universal set of the set of one-digit numbers, A be the set of even numbers, B be the set of prime numbers less than 10. Find the following:

a. $A - B$ or $A \backslash B$

b. $B - A$ or $B \backslash A$

c. $A \cup B$

d. $U - (A \cup B)$ or $U \backslash (A \cup B)$

Solution:

Here, $A = \{0, 2, 4, 6, 8\}, B = \{2, 3, 5, 7\}$. Then, we illustrate the sets using a Venn diagram as follows. From the Venn diagram we observe:

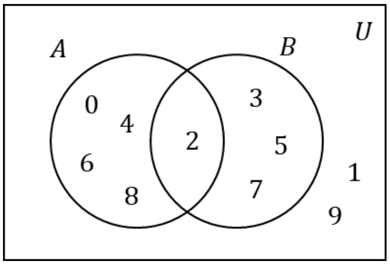


Figure 1.13

a. $A - B = \{0, 4, 6, 8\}$

b. $B - A = \{3, 5, 7\}$

c. $A \cup B = \{0, 2, 3, 4, 5, 6, 7, 8\}$

d. $U - (A \cup B) = \{1, 9\}$

Example 3

For the same sets in Example 2, find the following. What can you say from Example 2 a. and b.? What about d. and Example 2, a.?

a. A'

b. $U - A$

c. B'

d. $A \cap B'$

Solution:

a. $A' = \{1, 3, 5, 7, 9\}$

b. $U - A = \{1, 3, 5, 7, 9\}$

c. $B' = \{0, 1, 4, 6, 8, 9\}$

d. $A \cap B' = \{0, 4, 6, 8\}$

From a. and b., we can say, $A' = U - A$. From d. and Example 2, a., we can say, $A - B = A \cap B'$.

Theorem 1.1

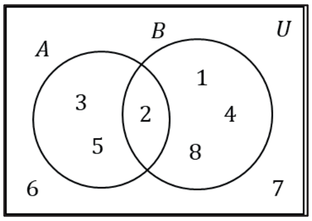
For any two sets A and B, each of the following holds true.

$(A')' = A$ $\quad$ $A' = U - A$

$A - B = A \cap B'$ $\quad$ $A \subseteq B \Leftrightarrow B' \subseteq A'$

Exercise 1.7

From the given Venn diagram, find each of the following:



a. $A - B$ or $A \backslash B$ $\quad$ **b.** $B - A$ or $B \backslash A$

c. $A \cup B$ $\quad$ d. $U - (A \cup B)$ or $U \backslash (A \cup B)$

Symmetric Difference of Two Sets

Definition 1.11 Symmetric Difference

For two sets A and B, the symmetric difference between these two sets is denoted by $A\Delta B$ and is defined as:

$A\Delta B = (A\backslash B) \cup (B\backslash A)$, which is $(A - B) \cup (B - A)$

or

$= (A \cup B)\backslash(A \cap B)$

In the Venn diagram, the shaded part represents $A\Delta B$

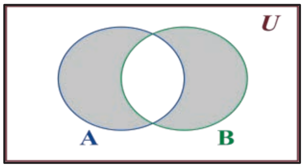


Figure 1.14

Example 1

Consider sets $A = \{1, 2, 4, 5, 8\}$ and $B = \{2, 3, 5, 7\}$.

Then, find $A\Delta B$.

Solution:

First, let us find $A\backslash B = \{1, 4, 8\}$ and $B\backslash A = \{3, 7\}$.

Hence, $A\Delta B = (A\backslash B) \cup (B\backslash A)$

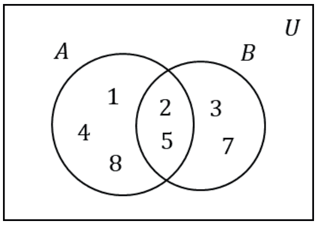


Figure 1.15

$= \{1, 4, 8\} \cup \{3, 7\} = \{1, 3, 4, 7, 8\}.$

Or $A\Delta B = (A \cup B)\backslash(A \cap B) = \{1, 3, 4, 7, 8\}.$

Example 2

Given sets $A = \{d, e, f\}$ and $B = \{4, 5, 6\}$. Then, find $A\Delta B$.

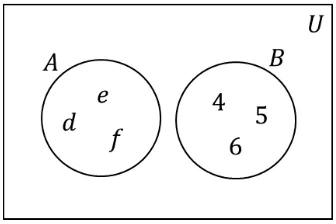


Figure 1.16

Solution:

First, we find $A\backslash B = \{d, e, f\}$ and $B\backslash A = \{4,5,6\}$.

Hence, $A\Delta B = (A\backslash B) \cup (B\backslash A)$

$= \{d, e, f\} \cup \{4, 5, 6\} = \{d, e, f, 4, 5, 6\}.$

Exercise 1.8

1. Given $A = \{0, 2, 3, 4, 5\}$ and $B = \{0, 1, 2, 3, 5, 7, 9\}$. Then, find $A\Delta B$.
2. If $A\Delta B = \emptyset$, then what can be said about the two sets?
3. For any two sets A and B, can we generalize $A\Delta B = B\Delta A$? Justify your answer.

Cartesian Product of Two Sets

Definition 1.12 Cartesian Product of Two Sets

The Cartesian product of two sets A and B, denoted by $A \times B$, is the set of all ordered pairs (a, b) where $a \in A$ and $b \in B$. This also can be expressed as $A \times B = \{(a, b): a \in A \text{ and } b \in B\}$.

Example 1

Let $A = \{1, 2\}$ and $B = \{a, b\}$. Then, find **a)** $A \times B$ **b)** $B \times A$

Solution:

a. $A \times B = \{(1, a), (1, b), (2, a), (2, b)\}$

b. $B \times A = \{(a, 1), (a, 2), (b, 1), (b, 2)\}$

Example 2

If $A \times B = \{(1,a),(1,b),(2,a),(2,b),(3,a),(3,b)\}$, then find sets A and B.

Solution:

A is the set of all first components of $A \times B$, that is, $A = \{1,2,3\}$, and

B is the set of all second components of $A \times B$, that is, $B = \{a,b\}$.

Exercise 1.9

1. Let $A = \{1,2,3\}$ and $B = \{e,f\}$. Then, find a) $A \times B$ b) $B \times A$.
2. If $A \times B = \{(7,6),(7,4),(5,4),(5,6),(1,4),(1,6)\}$, then find sets A and B.
3. If $A = \{a,b,c\}$, $B = \{1,2,3\}$ and $C = \{3,4\}$, then find $A \times (B \cup C)$.
4. If $A = \{6,9,11\}$, then find $A \times A$.
5. If the number of elements of set A is 6 and the number of elements of set B is 4, then the number of elements of $A \times B$ is ______________.

1.5 Application

Number of Elements of union of two sets

For the two subsets A and B of a universal set U, the following formula on the number of elements holds. That is

$$n(A \cup B) = n(A) + n(B) - n(A \cap B).$$

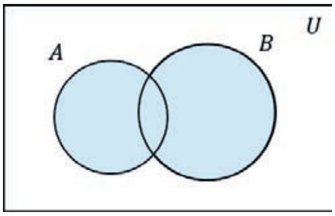


Figure 1.17

Especially in the case $A \cap B = \emptyset$, thus,

$n(A \cap B) = 0$, and the following holds:

$n(A \cup B) = n(A) + n(B)$.

If $A \cap B \neq \emptyset$, then

$n(A \cup B) = n(A) + n(B) - n(A \cap B)$.

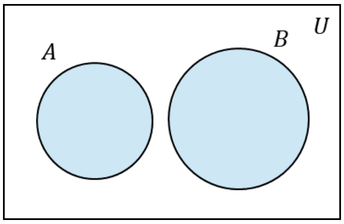


Figure 1.18

Example

Let A and B be two finite sets such that $n(A) = 20$, $n(B) = 28$, and $n(A \cup B) = 36$, then find $n(A \cap B)$.

Solution:

Using the formula $n(A \cup B) = n(A) + n(B) - n(A \cap B)$ we have

$$36 = 20 + 28 - n(A \cap B).$$

This gives $n(A \cap B) = (20 + 28) - 36 = 48 - 36 = 12.$

Exercise 1.10

1. Let A and B be two finite sets such that $n(A) = 34$, $n(B) = 46$ and $n(A \cup B) = 70$. Then, find $n(A \cap B)$.
2. There are 60 people attending a meeting. 42 of them drink tea and 27 drink coffee. If every person in the meeting drinks at least one of the two drinks, find the number of people who drink both tea and coffee. (Hint: Use a Venn diagram).

Summary

1. A set is a collection of well-defined objects or elements. When we say a set is well-defined, we mean that given an object we are able to determine whether the object is in the set or not.
2. Sets can be described in the following ways:

   Verbal method (Statement form)

   In this method, the well-defined description of the elements of the set is written in an ordinary English language statement form (in words).

   Complete listing method (Roster Method)

   In this method all the elements of the sets are completely listed. The elements are separated by commas and are enclosed within set brace, { }.

   Partial listing method

   We use this method, if listing of all elements of a set is difficult or impossible but the elements can be indicated clearly by listing a few of them that fully describe the set.

   Set builder method (Method of defining property)

   The set-builder method is described by a property that its member must satisfy. This is the method of writing the condition to be satisfied by a set or property of a set.
3. A set which does not contain any element is called an empty set, void set or null set. The empty set is denoted mathematically by the symbol { } or Ø.
4. Two sets A and B are said to be equal if and only if they have exactly same or identical elements. Mathematically, we write this as $A = B$.
5. Set A is said to be a subset of set B if every element of A is also an element of B. Mathematically, we write this as $A \subseteq B$.
   - Any set is a subset to itself.
   - Empty set is a sub set of every set.

- If set A is finite with n elements, then the number of subsets of set A is 2^n.

6. If $A \subseteq B$ and $A \neq B$, then A is called the proper subset of B and it can be written as $A \subset B$.

- For any set A, A is not a proper subset to itself.
- If set A is finite with n elements the number of proper subsets of set A is $2^n - 1$.
- Empty set is a proper subset of any other sets.

7. A universal set (usually denoted by U) is a set which has elements of all the related sets, without any repetition of elements.

8. The union of two sets A and B, which is denoted by $A \cup B$, is the set of all elements that are either in set A or in set B (or in both sets). We write this mathematically as

$$A \cup B = \{x \mid x \in A \text{ or } x \in B\}.$$

9. The intersection of two sets A and B, which is denoted by $A \cap B$, is the set of all elements that are in set A and in set B. We write this mathematically as

$$A \cap B = \{x \mid x \in A \text{ and } x \in B\}.$$

10. The difference between two sets A and B, which is denoted by $A - B$, is the set of all elements in set A and not in set B; this set is also called the relative complement of set A with respect to set B. We write this mathematically as

$$A - B = A \backslash B = \{x \mid x \in A \text{ and } x \notin B\}.$$

11. Let A be subset of a universal set U. The absolute complement (or simply complement) of set A, which is denoted by A', is defined as the set of all elements of U that are not in A. We write this mathematically as

$$A' = \{x \mid x \in U \text{ and } x \notin A\}.$$

12. A Venn diagram is a schematic or pictorial representation of the sets involved in the discussion. Usually sets are represented as interlocking circles, each of which is enclosed in a rectangle, which represents the universal set.

13. For two sets A and B the symmetric difference between these two sets is denoted by $A\Delta B$ and is defined as $A\Delta B = (A\backslash B) \cup (B\backslash A) = A \cup B)\backslash(A \cap B)$.

14. For any two finite sets A and B, $n(A \cup B) = n(A) + n(B) - n(A \cap B)$.

Review Exercise

1. Express following sets using the listing method.

a. A is the set of positive factors of 18

b. B is the set of positive even numbers below or equal to 30

c. $C = \{2n \mid n = 0, 1, 2, 3, \dots\}$

d. $D = \{x \mid x^2 = 9\}$

2. Express following sets using the set-builder method.

a. $\{2, 4, 6, \dots\dots\}$

b. $\{1, 3, 5, \dots, 99\}$

c. $\{1, 4, 9, \dots, 81\}$

3. Find all the subsets of the following sets.

a. $\{3, 4, 5\}$

b. $\{a, b\}$

4. Find $A \cup B$ and $A \cap B$ of the following.

a. $A = \{2, 3, 5, 7, 11\}$ and $B = \{1, 3, 5, 8, 11\}$

b. $A = \{x \mid x \text{ is the factor of } 12\}$ and $B = \{x \mid x \text{ is the factor of } 18\}$

c. $A = \{3x \mid x \in \mathbb{N}, x \leq 20\}$ and $B = \{4x \mid x \in \mathbb{N}, x \leq 15\}$

5. If $B \subseteq A, A \cap B' = \{1, 4, 5\}$, and $A \cup B = \{1, 2, 3, 4, 5, 6\}$, then find set B.

6. Let $A = \{2, 4, 6, 7, 8, 9\}$, $B = \{1, 3, 5, 6, 10\}$ and

$C = \{x \mid \in \mathbb{Z}, 3x + 6 = 0 \text{ or } 2x + 6 = 0\}$.

Find **a)** $A \cup B$

b) Is $(A \cup B) \cup C = A \cup (B \cup C)$?

7. Suppose the universal set U be the set of one-digit numbers, and set $A = \{x \mid x$ is an even natural number less than or equal to $9\}$.

Describe each set by complete listing method:

a. A'

b. $A \cap A'$

c. $A \cup A'$

d. $(A')'$

e. $\phi \setminus U$

f. ϕ'

g. U'

8. Let $U = \{1, 2, 3, 4, \ldots \ldots, 10\}$, $A = \{1, 3, 5, 7\}$, $B = \{1, 2, 3, 4\}$. Evaluate $U \setminus (A \Delta B)$.

9. Consider a universal set $U = \{1, 2, 3, \ldots, 14\}$, $A = \{2, 3, 5, 7, 11\}$, $B = \{2, 4, 8, 9, 10, 11\}$. Then, which one of the following is true?

A. $(A \cup B)' = \{1, 4, 6, 12, 13, 14\}$

B. $A \cap B = A' \cup B'$

C. $A \Delta B = (A \cap B)'$

D. $A \setminus B = \{3, 5, 7\}$

E. None

10. Let $A = \{3, 7, a^2\}$ and $B = \{2, 4, a + 1, a + b\}$ be two sets and all the elements of the two sets are integers. If $A \cap B = \{4, 7\}$, then find a and b. In addition, find $A \cup B$.

11. In a survey of 200 students in Motta Secondary School, 90 students are members of Nature club, 31 students are members of Mini-media club, 21 students are members of both clubs. Answer the following questions.

a. How many students are members of either of the clubs?

b. How many students are not members of either of the clubs?

c. How many students are only in Nature club?

12. A survey was conducted in a class of 100 children and it was found out that 45 of them like Mathematics whereas only 35 like Science and 10 students like both subjects. How many like neither of the subjects?

A) 70 B) 30 C) 100 D) 40

UNIT 2

THE NUMBER SYSTEM

Unit Outcomes

By the end of this unit, you will be able to:

- **Describe rational numbers.**
- **Locate rational numbers on number line.**
- **Describe irrational numbers.**
- **Locate some irrational numbers on a number line.**
- **Define real numbers.**
- **Classify real numbers as rational and irrational.**
- **Solve mathematical problems involving real numbers.**

Unit Contents

KEY Words

- division algorithm
- fundamental theorem of arithmetic
- rationalizing factor
- irrational number
- rationalization
- perfect square
- greatest common factor
- principal n^{th} root
- prime factorization
- repeating decimal
- scientific notation
- bar notation
- significant digits
- significant figures
- terminating decimal
- least common multiple (LCM)
- real number
- radicand
- composite number

Introduction

In the previous grades, you learned number systems about natural numbers, integers and rational numbers. You have discussed meaning of natural numbers, integers and rational numbers, the basic properties and operations on the above number systems. In this unit, after revising those properties of natural numbers, integers and rational numbers, you will continue to learn about irrational and real numbers.

2.1 Revision on Natural Numbers and Integers

Activity 2.1

1. List five members of :-
   a. Natural numbers
   b. Integers
2. Select natural numbers and integers from the following.
   a. 6 b. 0 c. -25
3. What is the relationship between natural numbers and integers?

4. Decide if the following statements are **always true, sometimes true or never true** and provide your justification.
   a. Natural numbers are integers
   b. Integers are natural numbers
   c. -7 is a natural number
5. Draw diagram which shows the relationship of Natural numbers and Integers.

The collection of well-defined distinct objects is known as a set. The word well-defined refers to a specific property which makes it easy to identify whether the given object belongs to the set or not. The word 'distinct' means that the objects of a set must be all different.

From your grade 7 mathematics lessons, you recall that

- The set of natural numbers, which is denoted by $\mathbb{N}$ expressed as
  $\mathbb{N} = \{1, 2, 3, \ldots\}$.
- The set of integers, which is denoted by $\mathbb{Z}$ is expressed as
  $\mathbb{Z} = \{\ldots, -2, -1, 0, 1, 2, 3, \ldots\}$.

Example

Categorize each of the following as natural numbers and integers.

$$3, -2, 11, 0, -18, 15, 7$$

Solution:

All the given numbers are integers and 3, 11, 15 and 7 are natural numbers.

Exercise 2.1

1. Categorize each of the following as natural numbers and integers.
   $8, -11, 23, 534, 0, -46, -19, 100$
2. What is the last integer before one thousand?

3. Consider any two natural numbers n_1 and n_2.
   a. Is $n_1 + n_2$ a natural number? Explain using example.
   b. Is $n_1 - n_2$ a natural number? Explain using example.
   c. What can you conclude from (a) and (b)?
4. If the perimeter of a triangle is 10 and lengths of the sides are natural numbers, find all the possible lengths of sides of the triangle.
5. Assume m and n are two positive integers and $m + n < 10$. How many different values can the product mn (m multiplied by n) have?

2.1.1 Euclid's Division lemma

Activity 2.2

1. In a book store there are 115 different books to be distributed to 8 students. If the book store shares these books equally, how many books will each student receive and how many books will be left?
2. Divide a natural number 128 by 6. What is the quotient and remainder of this process? Can you guess a remainder before performing the division process?

From your activity, using the process of dividing one positive integer by another, you will get remainder and quotient as described in the following theorem.

Theorem 2.1 Euclid's Division lemma

Given a non-negative integer a and a positive integer b, there exist unique non-negative integers q and r satisfying

$$a = b \times q + r \text{ with } 0 \leq r < b.$$

In theorem 2.1, a is called the dividend, q is called the quotient, b is called the divisor, and r is called the remainder.

Example

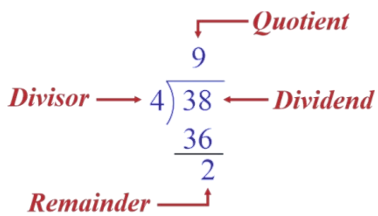
Find the unique quotient and remainder when a positive integer.

a. 38 is divided by 4 **b.** 5 is divided by 14

c. 12 is divided by 3 **d.** 2,574 is divided by 8

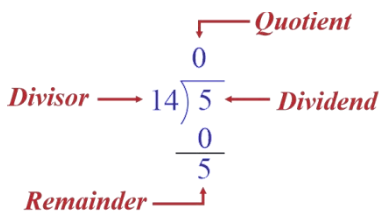
Solution:

a. Here, it is given that the dividend is $a = 38$ and the divisor is $= 4$. So that we need to determine the unique numbers q and r.When we divide 38 by 4, we get a quotient $q = 9$ and a remainder $r = 2$. Hence, we can write this as $38 = 4 \times 9 + 2$.



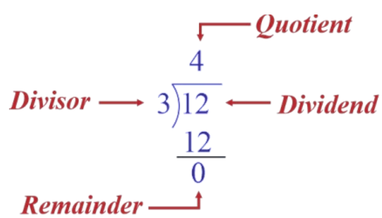
b. The number 5 is less than the divisor 14. So, the quotient is 0 and the remainder is 5.

That is, $5 = 14 \times 0 + 5$.

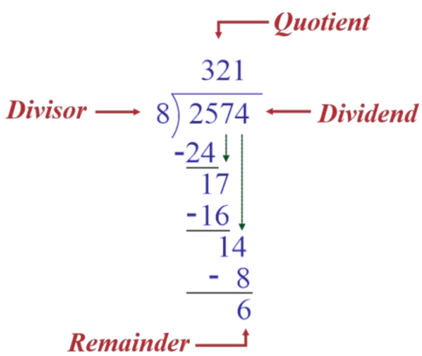


c. When we divide 12 by 3, we obtain 4 as a quotient and the remainder is 0. That is

$$12 = 3 \times 4 + 0.$$



d. Using long division if 2,574 is divided by 8, we get 321 as a quotient and 6 as a remainder. We can write $2{,}574 = 8 \times 321 + 6$.



Note

For two positive integers a and b in the division lemma, we say a is divisible by b if the remainder r is zero.

In the above example (c), 12 is divisible by 3 since the remainder is 0.

Exercise 2.2

1. For each of the following pairs of numbers, let a be the first number of the pair and b be the second number. Find q and r for each pair such that $a = b \times q + r$, where $0 \leq r < b$.

   a. $14, 3$ b. $116, 7$ c. $570, 6$ d. $25, 36$ e. $987, 16$

2. Find the unique quotient and remainder when 31 is divided by 6.
3. Find four positive integers when divided by 4 leaves remainder 3.
4. A man has Birr 68. He plans to buy items such that each costs Birr 7. If he needs Birr 5 to remain in his pocket, what is the maximum number of items he can buy?
5. Find the remainder of $\frac{(5m+1)(5m+3)(5m+6)}{5}$ for m is non-negative integer.

2.1.2 Prime numbers and composite numbers

In this subsection, you will confirm important facts about prime and composite numbers. The following activity (activity 2.3) will help you to refresh your memory.

Activity 2.3

1. Fill in the blanks to make the statements correct using the numbers 3 and 12.

   a. ____________is a factor of _____________.

   b. ____________is divisible by ____________.

   c. ____________is a multiple of _____________.

2. For each of the following statements write 'true' if the statement is correct and 'false' otherwise. If your answer is false give justification why it is false.
   a. 1 is a factor of all natural numbers.
   b. There is no even prime number.
   c. 23 is a prime number.
   d. If a number is natural number, it is either prime or composite.
   e. 351 is divisible by 3.
   f. $2^2 \times 3 \times 7$ is the prime factorization of 84.
   g. 63 is a multiple of 21.
3. Write factors of :
   a. 7 b. 15

Observations

Given two natural numbers h and p, h is called a multiple of p if there is a natural number q such that $h = p \times q$. In this way we can say:

- p is called a factor or a divisor of h.
- h is divisible by p.
- q is also a factor or divisor of h.
- h is divisible by q.

Hence, for any two natural numbers h and p, h is divisible by p if there exists a natural number q such that $h = p \times q$.

Definition 2.1 Prime and composite numbers

A natural number that has exactly two distinct factors, namely 1 (one) and itself is called a **prime number** whereas a natural number that has more than two factors is called a **composite number**.

Example 1

Is 18 a prime number or a composite number? Why?

Solution:

Observe that, $18 = 1 \times 18$, $18 = 2 \times 9$ or $18 = 3 \times 6$ .This indicates $1, 2, 3, 6, 9$ and 18 are factors of 18. Hence, 18 is a composite number.

Example 2

Find a prime number(s) greater than 50 and less than 55.

Solution:

The natural numbers greater than 50 and less than 55 are $51, 52, 53$ and 54.

$51 = 3 \times 17$, so that 3 and 17 are factors of 51 .

$52 = 2 \times 26$, so that 2 and 26 are factors of 52, and

$54 = 2 \times 27$, hence 2 and 27 are factors of 54. Therefore, these three numbers are composite numbers. But $53 = 1 \times 53$. Hence, 1 and 53 are the only factors of 53 so that 53 is a prime number. Therefore, 53 is a prime number greater than 50 and less than 55.

Exercise 2.3

1. Is 21 a prime number or a composite number? Why?
2. Write true if the statement is correct and false otherwise
   a. There are 7 prime numbers between 1 and 20.
   b. $4, 6, 15$ and 21 are composite numbers.
   c. The smallest composite number is 2.
   d. 101 is the prime number nearest to 100.
   e. No prime number greater than 5 ends with 5.
3. Is 28 a composite number? If so, list all of its factors.
4. Can we generalize that 'if a number is odd, then it is prime'? Why?

5. Which one of the following is true about composite numbers?

   A. They have 3 pairs of factors B. They are always prime numbers

   C. They do not have factors D. They have more than two factors

Note

- 1 is neither prime nor composite.
- 2 is the only even prime number.
- Factors of a number are always less than or equal to the number.

2.1.3 Divisibility test

In the previous lesson, you practiced how to get the quotient and remainder while you divide a positive integer by another positive integer. Now you will revise the divisibility test to check whether a is divisible by b or not without performing division algorithm.

Activity 2.4

Check whether the first integer is divisible by the second or not without using division algorithm.

a. $2584, 2$ b. $765, 9$ c. $63885, 6$ d. $7964, 4$ e. $65475, 5$

From your activity, you may check this either by using division algorithm or by applying rules without division.

Every number is divisible by 1. You need to perform the division procedure to check divisibility of one natural number by another. The following rules can help you to determine whether a number is divisible by $2, 3, 4, 5, 6, 8, 9$ and 10. Divisibility test by 7 will not be discussed now because it is beyond the scope of this level.

Divisibility test:- It is an easy way to check whether a given number is divisible by $2, 3, 4, 5, 6, 8, 9$ or 10 without actually performing the division process.

A number is divisible by

- 2, if its unit digit is divisible by 2.
- 3, if the sum of its digits is divisible by 3.
- 4, if the number formed by its last two digits is divisible by 4.
- 5, if its unit digit is either 0 or 5.
- 6, if it is divisible by 2 and 3.
- 8, if the number formed by its last three digits is divisible by 8.
- 9, if the sum of its digits is divisible by 9.
- 10, if its unit digit is 0.

Example

Using divisibility test check whether 2,334 is divisible by 2, 3, 4, 5, 6, 8, 9 and 10.

Solution:

- 2,334 is divisible by 2 since its unit digit 4 is divisible by 2.
- 2,334 is divisible by 3 since the sum of the digits $(2 + 3 + 3 + 4)$ is 12 and it is divisible by 3.
- 2,334 is not divisible by 4 since its last two digits, that is 34 is not divisible by 4.
- 2,334 is not divisible by 5 since its unit digit is not either 0 or 5.
- 2,334 is divisible by 6 since it is divisible by 2 and 3.
- 2,334 is not divisible by 8 since its last three digits 334 is not divisible by 8.
- 2,334 is not divisible by 9 since the sum of the digits $(2 + 3 + 3 + 4 = 12)$ is not divisible by 9.
- 2,334 is not divisible by 10 since its unit digit is not zero.

Exercise 2.4

1. Using divisibility test, check whether the following numbers are divisible by

2, 3, 4, 5, 6, 8, 9 and 10:

a. 384 b. 3,186 c. 42,435

2. Given that $74{,}3x2$ is a number where x is its tens place. If this number is divisible by 8, what is (are) the possible value(s) of x?
3. Find the least possible value of the blank space so that the number 3457__40 is divisible by 4.
4. Fill the blank space with the smallest possible digit that makes the given number 81231_37 is divisible by 9.

Definition 2.2 Prime factorization

The expression of a composite number as a product of prime numbers is called prime factorization.

Consider a composite number which we need to write it in prime factorized form. Recall that a composite number has more than two factors. The factors could be prime or still composite. If both of the factors are prime, we stop the process by writing the given number as a product of these prime numbers. If one of the factors is a composite number, we will continue to get factors of this composite number till all the factors become primes. Finally, express the number as a product of all primes which are part of the process.

Example 1

Express each of the following numbers as prime factorization form.

a. 6 **b.** 30 **c.** 72

Solution:

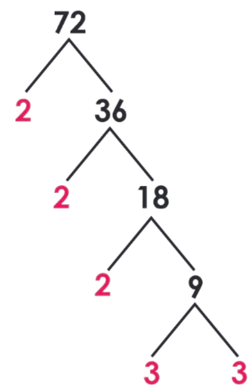
a. $6 = 2 \times 3$, since both 2 and 3 are prime, we stop the process.

b. $30 = 2 \times 15$ and $15 = 3 \times 5$.

So that, $30 = 2 \times 3 \times 5$. The right hand side of this equation is the prime factorization of 30.

The process of prime factorization can be easily visualized by using a factor tree as shown below.

c. By divisibility test, 72 is divisible by 2. So that 2 is one of its factors and it will be written as 2×36 as shown in the right side. Again 36 is divisible by 2. Hence, $36 = 2 \times 18$ and this process will continue till we get the factors as primes. Hence, the prime factorization of 72 is given as



$72 = 2 \times 2 \times 2 \times 3 \times 3 = 2^3 \times 3^2$. Also we can observe

$72 = 8 \times 9 = 2 \times 2 \times 2 \times 3 \times 3 = 2^3 \times 3^2$

$72 = 4 \times 18 = 2 \times 2 \times 2 \times 3 \times 3 = 2^3 \times 3^2$

$72 = 3 \times 24 = 3 \times 3 \times 2 \times 2 \times 2 = 2^3 \times 3^2$.

The following theorem is stated without proof to generalize factors of composite numbers.

Theorem 2.2 Fundamental theorem of arithmetic

Every composite number can be expressed (factored) as a product of primes and this factorization is unique.

Example 2

Express 456 as prime factorization form.

Solution:

To determine the prime factors, we will divide 456 by 2. If the quotient is also a composite number, again using a divisibility test we will search the prime number which divides the given composite number. Repeat this process till the quotient is prime.

That is, $456 \div 2 = 228$

$228 \div 2 = 114$

$114 \div 2 = 57$

$57 \div 3 = 19$, we stop the process since 19 is prime.

Hence, $456 = 2^3 \times 3 \times 19$.

Exercise 2.5

1. Express each of the following numbers as prime factorization form.
   a. 21 b. 70 c. 105
   d. 252 e. 360 f. 1,848
2. 180 can be written as $2^a \times 3^b \times 5^c$. Then, find the value of a, b and c.
3. Find the four prime numbers whose product is 462.

2.1.4 Greatest common factor and least common multiple

In this subsection, you will revise the basic concepts about greatest common factors and least common multiples of two or more natural numbers.

Greatest common factor

Activity 2.5

1. Given the numbers 12 and 16:
   a. Find the common factors of the two numbers.
   b. Find the greatest common factor of the two numbers.
2. Given the following three numbers 24, 42 and 56:
   a. Find the common factors of the three numbers.
   b. Find the greatest common factor of the three numbers.

Definition 2.3

i) Given two or more natural numbers, a number which is a factor of these natural numbers is called a common factor.

ii) The Greatest Common Factor (GCF) or Highest Common Factor (HCF) of two or more natural numbers is the greatest natural number of the common factors. $\text{GCF}(a, b)$ is to mean the Greatest Common Factor of a and b.

Example 1

Find the greatest common factors of 36 and 56.

Solution:

Factors of 36 are $1, 2, 3, 4, 6, 9, 12, 18, 36$ and factors of 56 are $1, 2, 4, 7, 8, 14, 28, 56$. Here 1,2 and 4 are the common factors of both 36 and 56. The greatest one from these common factors is 4. Hence $\text{GCF}(36, 56) = 4$. You can also observe this from the following Venn diagram.

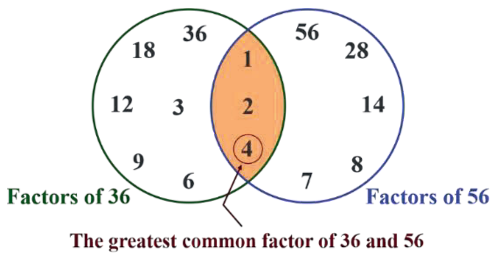


Figure 2.1 Factors of 36 and 56

Activity 2.6

Let $p = 36, q = 56$. Then, write

a. The prime factorization of p and q.

b. Write common prime factors of p and q with the least power.

c. Take the product of the common prime factors you found in the above (b) if they are two or more.

d. Compare your result in (c) with $\text{GCF}(36, 56)$ you got in example 1 above.

The above activity leads you to use another alternate approach of determining GCF of two or more natural numbers using prime factorization. Using this method, the GCF of two or more natural numbers is the product of their common prime factors and the smallest number of times each power appears in the prime factorization of the given numbers.

Example 2

Use prime factorization to find $\text{GCF}(12, 18, 36)$.

Solution:

Let us write each of the given three numbers as prime factorization form.

$$12 = 2^2 \times 3^1$$
$$18 = 2^1 \times 3^2$$
$$36 = 2^2 \times 3^2$$

In the above factorizations, 2 and 3 are common prime factors of the numbers (12,18 and 36). Further the least power of 2 is 1 and least power of 3 is also 1. So that the product of these two common prime numbers with least power of each is $2 \times 3 = 6$. Hence, $\text{GCF}(12, 18, 36) = 6$.

Exercise 2.6

Find the greatest common factors (GCF) of the following numbers.

1. Using Venn diagram method
   a. $12, 18$
   b. $24, 64$
   c. $45, 63, 99$
2. Using prime factorization method
   a. $24, 54$
   b. $108, 104$
   c. $180, 270, 1{,}080$

Least common multiple

Activity 2.7

For this activity, you need to use pen and pencil.

- Write the natural numbers from 1 to 60 using your pen.
- Encircle the number from the list which is a multiple of 6 using pencil.
- Underline the number on the list which is a multiple of 8 using pencil.

Using the above task:

a. Collect the numbers from the list which are both encircled and underlined in a set.

b. What is the least common number from the set you found in (a)?

c. What do you call the number you get in (b) above for the two numbers 6

Definition 2.4

For any two natural numbers a and b, the Least Common Multiple of a and b denoted by $\text{LCM}(a, b)$ is the smallest multiple of both a and b.

Intersection Method

Example 1

Find Least Common Multiple of 2 and 3, that is, $\text{LCM}(2, 3)$.

Solution:

$2, 4, \mathbf{6}, 8, 10, \mathbf{12}, 14, 16, \ldots$ are multiples of 2 and $3, \mathbf{6}, 9, \mathbf{12}, 15, \mathbf{18}, \ldots$ are multiples of 3. Hence, $\mathbf{6}, \mathbf{12}, \mathbf{18}, \ldots$ are common multiples of 2 and 3. The least number from the common multiples is 6.

Therefore, $\text{LCM}(2, 3)$ is 6.

Example 2

Find Least Common Multiple of 6 and 9, that is, $\text{LCM}(6, 9)$.

Solution:

$6, 12, \mathbf{18}, 24, 30, \mathbf{36}...$ are multiples of 6 and $9, \mathbf{18}, 27, \mathbf{36}, 45, ...$ are multiples of 9. Hence, $\mathbf{18}, \mathbf{36}, \mathbf{54}, ...$ are common multiples of 6 and 9. The least number from the common multiples is 18.

Therefore, $\text{LCM}(6, 9)$ is 18.

Factorization Method

Example 3

Find the Least Common Multiple of 2, 3 and 5, that is, $\text{LCM}(2, 3, 5)$ using factorization method.

Solution:

2,3 and 5 are prime numbers. Taking the product of these prime numbers gives us $\text{LCM}(2,3,5) = 2 \times 3 \times 5 = 30$.

Example 4

Find the Least Common Multiple of 6,10 and 16, that is, $\text{LCM}(6, 10, 16)$ using factorization method.

Solution:

Writing each by prime factorization, we have

$$\left.\begin{aligned} 6 &= 2 \times 3 \\ 10 &= 2 \times 5 \\ 16 &= 2^4 \end{aligned}\right\}$$ The prime factors that appear in these factorization are 2, 3 and 5.

Taking the product of the highest powers gives us

$\text{LCM}(6, 10, 16) = 2^4 \times 3 \times 5 = 240$.

Activity 2.8

Consider two natural numbers 15 and 42. Then, find

a. $\text{LCM}(15,42)$ and $\text{GCF}(15, 42)$.
b. 15×42

c. What is the product of $GCF(15, 42)$ and $LCM(15, 42)$?

d. Compare your results of (b) and (c).

e. What do you generalize from (d)?

From the above activity you can deduce that

- For any two natural numbers a and b, $GCF(a, b) \times LCM(a, b) = ab$.

Example 5

Find $GCF(12, 18) \times LCM(12, 18)$.

Solution:

The product of $GCF(a, b)$ and $LCM(a, b)$ is the product of the two numbers a and b for any two natural numbers a and b. Hence

$GCF(12, 18) \times LCM(12, 18) = 12 \times 18 = 216$.

Exercise 2.7

1. Find the least common multiples (LCM) of the following list of numbers using both intersection method and prime factorization method.

   a. $6, 15$ b. $14, 21$ c. $4, 15, 21$ d. $6, 10, 15, 18$

2. Find GCF and LCM of each of the following numbers

   a. 4 and 9 b. 7 and 48 c. 12 and 32

   d. 16 and 39 e. $12, 16$ and 24 f. $4, 18$ and 30

3. Find $GCF(16, 24) \times LCM\ (16, 24)$.

4. In a school, the number of participants in Sport, Mini media and Anti Aids club are $60, 84$ and 108 respectively. Find the minimum number of rooms required. In each room the same number of participants are to be seated and all of them being in the same club.

2.2 Rational Numbers

In section 2.1, you have learnt about natural numbers and integers. In this section you will extend the set of integers to the set of rational numbers. You will also discuss how to represent rational numbers as decimals and locate them on the number line.

Activity 2.9

Given integers $7, -2, 6, 0$ and -3.

a. Divide one number by another (except division by 0).

b. From the result obtained in (a) which of them are integers?

c. What can you conclude from the above (a) and (b)?

Now let us discuss about rational numbers

Definition 2.5 Rational Numbers

Any number that can be written in the form $\frac{a}{b}$, where a and b are integers and $b \neq 0$ is called a rational number. The set of rational numbers is denoted by $\mathbb{Q}$ and is described as

$$\mathbb{Q} = \left\{\frac{a}{b} : a, b \in \mathbb{Z} \text{ and } b \neq 0\right\}.$$

Example 1

The numbers $\frac{3}{4}, -6, 0, \frac{-13}{10}$ are rational numbers. Here, -6 and 0 can be written as $\frac{-6}{1}$ and $\frac{0}{1}$, respectively. So we can conclude every natural number and integer is a rational number. You can see this relation using figure 2.2.

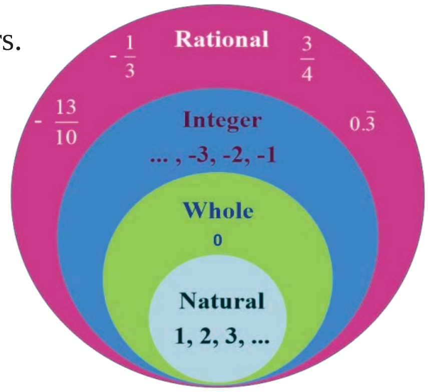


Figure 2.2

Note

Suppose $x = \frac{a}{b} \epsilon \mathbb{Q}$, x is a fraction with numerator a and denominator b,

i) If $a < b$, then x is called proper fraction.

ii) If $a \geq b$, then x is called improper fraction.

iii) If $y = c\frac{a}{b}$, where $c \epsilon \mathbb{Z}$ and $\frac{a}{b}$ is a proper fraction, then y is called a mixed fraction (mixed number).

iv) x is said to be in simplest (lowest form) if a and b are relatively prime or $\text{GCF}(a, b) = 1$.

Example 2

Categorize each of the following as proper, improper or mixed fraction

$\frac{7}{8}, \frac{2}{9}, 5\frac{1}{4}, \frac{7}{3}$ and 6.

Solution: $\frac{7}{8}$ and $\frac{2}{9}$ are proper fractions,

$\frac{7}{3}$ and 6 are improper fraction (since $6 = \frac{6}{1}$) and

$5\frac{1}{4}$ is a mixed fraction.

Example 3

Express $3\frac{1}{4}$ as improper fraction.

Solution: For three integers l, m, n where $n \neq 0$: $l\frac{m}{n} = l + \frac{m}{n} = \frac{(l \times n)+m}{n}$.

So that $3\frac{1}{4} = \frac{(3\times4)+1}{4} = \frac{13}{4}$.

Exercise 2.8

1. Express each of the following integers as fraction.

   a. 5 b. -3 c. 13

2. Write **true** if the statement is correct and **false** otherwise. Give justification if the answer is false.

a. Any integer is a rational number.

b. The simplest form of $\frac{35}{45}$ is $\frac{9}{7}$.

c. For $a, b \in \mathbb{Z}, \frac{a}{b}$ is a rational number.

d. The simplest form of $5\frac{2}{4}$ is $\frac{11}{2}$.

3. If $\frac{a}{b}$ and $\frac{c}{d}$ are two rational numbers, show that $\frac{a}{b} \times \frac{c}{d}$ is also a rational number.

4. Zebiba measures the length of a table and she reads 54 cm and 4 mm. Express this measurement in terms of cm in lowest form of $\frac{a}{b}$.

5. A rope of $5\frac{1}{3}$ meters is to be cut into 4 pieces of equal length. What will be the length of each piece?

2.2.1 Representation of rational numbers by decimals

In this subsection, you will learn how to represent rational numbers by decimals and locate the rational number on the number line.

Activity 2.10

1. Perform each of the following divisions.

a. $\frac{3}{5}$ b. $\frac{5}{9}$ c. $-\frac{14}{15}$ d. $\frac{-2}{7}$

2. Write the numbers 0.2 and 3.31 as a fraction form.

From the above activity 2.10, you may observe the following:

- Any rational number $\frac{a}{b}$ can be written as decimal form by dividing the numerator a by the denominator b.
- When we change a rational number $\frac{a}{b}$ into decimal form, one of the following cases will occur
  - The division process ends when a remainder of zero is obtained. Here, the decimal is called a terminating decimal.
  - The division process does not terminate but repeats as the remainder never become zero. In this case the decimal is called repeating decimal.

In the above activity 2.10 (1), when you perform the division $\frac{3}{5}$, you obtain a decimal 0.6 which terminates, $\frac{5}{9} = 0.555\ldots$ and $\frac{-14}{15} = -0.933\ldots$ are repeating.

Notation

- To represent repetition of digit/digits we put a **bar notation** above the repeating digit/digits.

Example 1

Write the following fractions using the notation of repeating decimals.

a. $\frac{1}{3}$ **b.** $\frac{23}{99}$

Solution:

a. $\frac{1}{3} = 0.333\ldots = 0.\overline{3}$ **b.** $\frac{23}{99} = 0.232323\ldots = 0.\overline{23}$.

Converting terminating decimals to fractions

Every terminating decimal can be written as a fraction form with a denominator of a power of 10 which could be 10, 100, 1000 and so on depending on the number of digits after a decimal point.

Example 2

Convert each of the following decimals to fraction form.

a. 7.3 **b.** -0.18

Solution:

a. The smallest place value of the digits in the number 7.3 is in the tenths column so we write this as a number of tenths. So that, $7.3 = \frac{7.3\times10}{10} = \frac{73}{10}$ (since it has 1 digit after a decimal point, multiply both the numerator and the denominator by 10).

b. The smallest place value of the digits in the number -0.18 is in the hundredths column so we write this as a number of hundredths.

Hence, $-0.18 = \frac{-0.18\times100}{100} = \frac{-18}{100}$ and after simplification, we obtain $\frac{-9}{50}$.

Exercise 2.9

1. Write the following fractions using the notation of repeating decimals.

   a. $\frac{8}{9}$ b. $\frac{1}{11}$ c. $\frac{5}{6}$ d. $\frac{14}{9}$

2. Convert each of the following as a fraction form.

   a. 0.3 b. 3.7 c. 0.77 d. −12.369

   e. 0.6 f. 9.5 g. −0.48 h. −32.125

3. Using long division, we get that $\frac{1}{7} = 0.\overline{142857}$. Can you predict the decimal expression of $\frac{2}{7}, \frac{3}{7}, \frac{4}{7}, \frac{5}{7}$ and $\frac{6}{7}$ without doing long division? If so, how?

4. What can you conclude from question no. 3 above?

Representing rational numbers on the number line

Example 1

Locate the rational numbers $-5, 3, \frac{4}{3}$ and $\frac{-5}{2}$ on the number line.

Solution:

You can easily locate the given integers −5 and 3 on a number line. But to locate a fraction, change the fraction into decimal. That is, $\frac{4}{3} = 1.\overline{3}$ and $\frac{-5}{2} = -2.5$. Now locate each of these by bold mark on the number line as shown in figure 2.3.

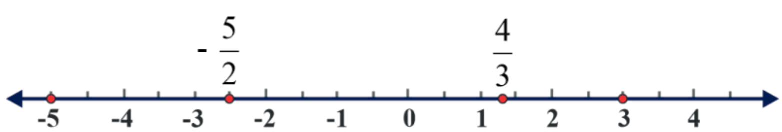


Figure 2.3

2.2.2 Conversion of repeating decimals into fractions

In section 2.2.1, you have discussed how to convert the terminating decimals into fractions. In this subsection, you will learn how repeating decimals can be converted to fractions.

Example 2

Represent each of the following decimals as a simplest fraction form (ratio of two integers).

a. $0.\overline{5}$ **b.** $2.\overline{12}$

Solution:

a. We need to write $0.\overline{5}$ as $\frac{a}{b}$ form. Now let $d = 0.\overline{5}$. Then,

$10d = 5.\overline{5}$ (multiplying both sides of $d = 0.\overline{5}$ by 10)

$-\ \ d = 0.\overline{5}$ (we use subtraction to eliminate the repeating part)

$9d = 5$.

Now dividing both sides by 9, we have $d = \frac{5}{9}$. Hence, the fraction form of $0.\overline{5}$ is $\frac{5}{9}$.

b. We need to write $2.\overline{12}$ as $\frac{a}{b}$ form. Now let $d = 2.\overline{12}$. Then,

$100d = 212.\overline{12}$ (multiplying both sides of $d = 2.\overline{12}$ by 10)

$-d = \quad 2.\overline{12}$ (we use subtraction to eliminate the repeating part)

$99d = 210$.

Now, dividing both sides by 99 results $d = \frac{210}{99} = \frac{70}{33}$. Hence, the fraction form of $2.\overline{12}$ is $\frac{70}{33}$.

Example 3

Represent each of the following in fraction form.

a. $0.1\overline{2}$ **b.** $1.2\overline{16}$

Solution:

a. Let $d = 0.\overline{1}2$, now

$100d = 12.\overline{2}$ (multiplying both sides of $d = 0.\overline{1}2$ by 100)

$\underline{-10d = 1.\overline{2}}$ (multiplying both sides of $d = 0.\overline{1}2$ by 10)

$90d = 11.$

Solving for d, that is dividing both sides by 90 we obtain $d = \frac{11}{90}$. Hence, the fraction form of $0.\overline{1}2$ is $\frac{11}{90}$.

b. Using similar procedure of (a), let $d = 1.\overline{2}16$. Now

$1000d = 1216.\overline{1}6$ (multiplying both sides of $d = 1216.\overline{1}6$ by 1000)

$\underline{-10d = 12.\overline{1}6}$ (multiplying both sides of $d = 12.\overline{1}6$ by 10)

$990d = 1204.$

Dividing both sides by 990, we obtain $d = \frac{1204}{990} = \frac{602}{495}$. Hence, the fraction form of $1.\overline{2}16$ is $\frac{602}{495}$.

Exercise 2.10

1. Locate the rational numbers $3, 4, \frac{4}{5}, 0.\overline{4}$ and $2\frac{1}{3}$ on the number line.
2. Represent each of the following decimals as simplest fraction form.

   a. $2.\overline{6}$ b. $0.\overline{1}4$ c. $0.\overline{7}16$ d. $1.3\overline{2}12$ e. $-0.53\overline{2}13$

2.3 Irrational Numbers

Recall that terminating and repeating decimals are rational numbers. In this subsection, you will learn about decimals which are neither terminating nor repeating

2.3.1 Neither repeating nor terminating numbers

Do you recall a perfect square?

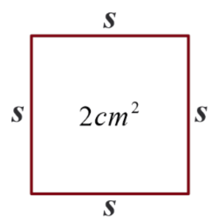
A perfect square is a number that can be expressed as a product of two equal integers. For instance, $1, 4, 9$ are perfect squares since $1 = 1^2, 4 = 2^2, 9 = 3^2$.

A square root of a number x is a number y such that $y^2 = x$. For instance, 5 and -5 are square roots of 25 because $5^2 = (-5)^2 = 25$.

Consider the following situation.

There is a square whose area is 2 cm^2. What is the length of a side of this square?

The length of a side is supposed to be s cm.



Since $1^2 = 1$, area of the square which side is 1 cm is 1 cm^2.

Likewise, $2^2 = 4$, area of the square whose side is 2 cm is 4 cm^2. From the above, $1 < s < 2$.

In a similar way,

$$1.4^2 < 2 < 1.5^2 \Rightarrow 1.4 < s < 1.5$$

$$1.41^2 < 2 < 1.42^2 \Rightarrow 1.41 < s < 1.42$$

$$1.414^2 < 2 < 1.415^2 \Rightarrow 1.414 < s < 1.415$$

Repeating the above method, we find $s = 1.414 \ldots$. This number is expressed as

$s = \sqrt{2}$, and $\sqrt{2} = 1.414 \ldots$. The sign $\sqrt{\ }$ is called a radical sign.

Example 1

Find the number without radical sign.

a. $\sqrt{4}$ **b.** $\sqrt{9}$ **c.** $-\sqrt{9}$ **d.** $\sqrt{0.0016}$

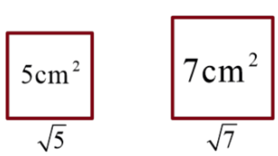
Solution:

a. $\sqrt{4} = 2$ since $2^2 = 4$ **b.** $\sqrt{9} = \sqrt{3^2} = 3 = \frac{3}{1}$

c. $-\sqrt{9} = -\sqrt{3^2} = -3$ **d.** $\sqrt{0.0016} = \sqrt{(0.04)^2} = 0.04 = \frac{4}{100} = \frac{1}{25}$

Consider the magnitude of $\sqrt{5}$ and $\sqrt{7}$. Which one is greater?

The area of square with side length $\sqrt{5}$ cm is 5 cm^2.



The area of square with side length $\sqrt{7}$ cm is 7 cm^2. From the figure, when the side length becomes greater, the area becomes greater, too.

Hence, $\sqrt{5} < \sqrt{7}$ since $5 < 7$. This leads to the following conclusion.

When $a > 0, b > 0$, if $a < b$, then $\sqrt{a} < \sqrt{b}$.

Example 2

Compare $\sqrt{5}$ and $\sqrt{6}$.

Solution:

Since $5 < 6$ then $\sqrt{5} < \sqrt{6}$.

Exercise 2.11

1. Find the numbers without radical sign.

   a. $\sqrt{25}$ b. $-\sqrt{36}$ c. $\sqrt{0.04}$ d. $-\sqrt{0.0081}$

2. Compare the following pairs of numbers.

   a. $\sqrt{7}, \sqrt{8}$ b. $\sqrt{3}, \sqrt{9}$ c. $\sqrt{0.04}, \sqrt{0.01}$ d. $-\sqrt{3}, -\sqrt{4}$

What can you say about the square root of a number which is not a perfect square?

To understand the nature of such numbers, you can use a scientific calculator and practice the following.

Let us try to determine $\sqrt{2}$ using a calculator. Press 2 and then the square root button.

A value for $\sqrt{2}$ will be displayed on the screen of the calculator as $\sqrt{2} \approx 1.4142135624$.

The calculator provides a terminated result due to its capacity (the number of digit you get may be different for different calculators). But this result is not a terminating or

Figure 2.4 Scientific calculator

repeating decimal. Hence $\sqrt{2}$ is not rational number. Similarly, check $\sqrt{3}$ and $\sqrt{7}$ are not terminating

and repeating. Such types of numbers are called irrational numbers.

Definition 2.6

A decimal number that is neither terminating nor repeating is an **irrational number**.

Remark

In general, if a is natural number that is not perfect square, then $\sqrt{a}$ is an irrational number.

Example

Determine whether each of the following numbers is rational or irrational.

a. $\sqrt{25}$ **b.** $\sqrt{0.09}$ **c.** 0.12345 ... **d.** 0.010110111 ... **e.** $\frac{22}{7}$ **f.** π

Solution:

a. $\sqrt{25} = \sqrt{5^2} = 5$ which is rational.

b. $\sqrt{0.09} = \sqrt{0.3^2} = 0.3$ which is rational number.

c. $0.12345\ldots$ is neither terminating nor repeating, so it is irrational.

d. $0.010110111\ldots$ is also neither repeating nor terminating, so it is irrational number.

e. $\frac{22}{7}$ is a fraction form so that it is rational.

f. π which is neither terminating nor repeating, so that it is irrational.

This example (c & d) leads you to the fact that, there are decimals which are neither repeating nor terminating.

Exercise 2.12

Determine whether each of the following numbers is rational or irrational.

a. $\sqrt{36}$ b. $\sqrt{7}$ c. $\frac{6}{5}$ d. $\sqrt{0.01}$

e. $-\sqrt{13}$ f. $\sqrt{11}$ g. $\sqrt{18}$

Locating irrational number on the number line

Given an irrational number of the form $\sqrt{a}$ where a is not perfect square. Can you locate such a number on the number line?

Example

Locate $\sqrt{2}$ on the number line.

Solution:

We know that 2 is a number between perfect squares 1 and 4. That is $1 < 2 < 4$. Take a square root of all these three numbers, that is, $\sqrt{1} < \sqrt{2} < \sqrt{4}$. Therefore $1 < \sqrt{2} < 2$. Hence, $\sqrt{2}$ is a number between 1 and 2 on the number line.

(You can also show $\sqrt{2} \approx 1.4142...$ using a calculator).

To locate $\sqrt{2}$ on the number line, you need a compass and straightedge ruler to perform the following.

- Draw a number line. Label an initial point 0 and points 1 unit long to the right and left of 0. Construct a perpendicular line segment 1 unit long at 1.

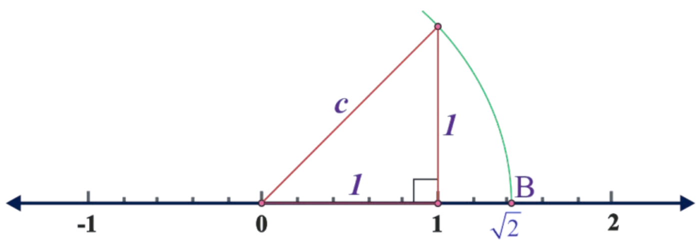


Figure 2.5 Location of $\sqrt{2}$ on the number line

- Draw a line segment from the point corresponding to 0 to the top of the 1unit segment and label its length as c.
- Using Pythagorean Theorem, $1^2 + 1^2 = c^2$ so that $c = \sqrt{2}$ unit long.
- Open the compass to the length of c. With the tip of the compass at the point corresponding to 0, draw an arc that intersects the number line at B. The distance from the point corresponding to 0 to B is $\sqrt{2}$ unit.

The following figure 2.6 could indicate how other irrational numbers are constructed on the number line using $\sqrt{2}$.

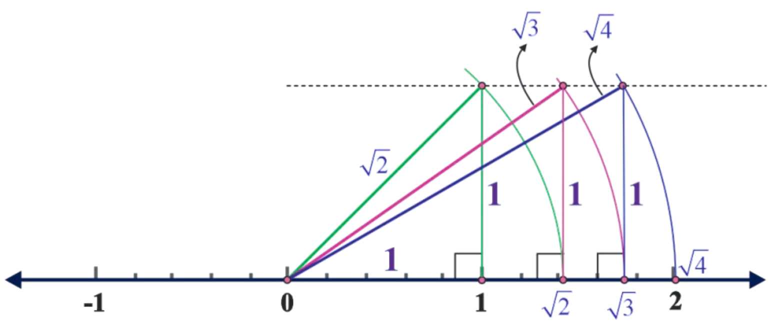


Figure 2.6

Exercise 2.13

1. Locate the following on the number line.

   a. $\sqrt{3}$ b. $\sqrt{5}$ c. $-\sqrt{3}$

2. Between which natural numbers are the following numbers?

   a. $\sqrt{3}$ b. $\sqrt{5}$ c. $\sqrt{6}$

3. Write 'True' if the statement is correct and 'False' otherwise.

   a. Every point on the number line is of the form $\sqrt{n}$, where n is a natural number.

   b. The square root of all positive integers is irrational.

2.3.2 Operations on irrational numbers

In section 2.3.1, you have seen what an irrational number is and how you represent it on the number line. In this section, we will discuss addition, subtraction, multiplication and division of irrational numbers.

Activity 2.11

1. Identify whether $1+\sqrt{2}$ is rational or irrational.
2. Is the product of two irrational numbers irrational? Justify your answer with examples.

When $a > 0, b > 0$, then $\sqrt{a} \times \sqrt{b} = \sqrt{ab}$.

Example

Calculate each of the following.

a. $\sqrt{2} \times \sqrt{3}$ **b.** $\frac{2}{\sqrt{3}} \times \frac{1}{\sqrt{3}}$ **c.** $(2+\sqrt{2}) \times (1+\sqrt{5})$ **d.** $(\sqrt{5}+\sqrt{3})^2$

Solutions:

a. $\sqrt{2} \times \sqrt{3} = \sqrt{2 \times 3} = \sqrt{6}$

b. $\frac{1}{\sqrt{3}} \times \frac{2}{\sqrt{3}} = \frac{1 \times 2}{\sqrt{3} \times \sqrt{3}} = \frac{2}{\sqrt{3 \times 3}} = \frac{2}{3}$

c. Using distribution of addition over multiplication

$$(2+\sqrt{2}) \times (1+\sqrt{5}) = (2 \times 1) + (2 \times \sqrt{5}) + (\sqrt{2} \times 1) + (\sqrt{2} \times \sqrt{5})$$

$$= 2 + 2\sqrt{5} + \sqrt{2} + \sqrt{10}$$

d. $(\sqrt{5}+\sqrt{3})^2 = (\sqrt{5}+\sqrt{3})(\sqrt{5}+\sqrt{3}) = (\sqrt{5})^2 + 2 \cdot \sqrt{5}\sqrt{3} + (\sqrt{3})^2$

$$= 8 + 2\sqrt{15}$$

Exercise 2.14

1. Calculate each of the following.

a. $\sqrt{3} \times \sqrt{5}$

b. $2\sqrt{5} \times \sqrt{7}$

c. $-\sqrt{2} \times \sqrt{6}$

d. $\frac{1}{\sqrt{5}} \times \frac{10}{\sqrt{5}}$

e. $(2+\sqrt{3}) \times (-2+\sqrt{3})$

f. $(\sqrt{3}+\sqrt{2})^2$

g. $(\sqrt{7}+\sqrt{3})(\sqrt{7}-\sqrt{3})$

h. $(\sqrt{6}-\sqrt{10})^2$

2. Decide whether the following statements are **always true, sometimes true or never true** and give your justification.

a. The product of two irrational numbers is rational.

b. The product of two irrational numbers is irrational.

c. Any irrational number can be written as a product of two irrational numbers.

d. Irrational numbers are closed with respect to multiplication.

Activity 2.12

1. Consider irrational numbers $\sqrt{2}$, $\sqrt{3}$, $\sqrt{5}$ and $\sqrt{8}$. Divide each of these numbers by $\sqrt{2}$.

2. What do you conclude from the result you obtained in (1) above.

When $a > 0, b > 0$, then $\frac{\sqrt{a}}{\sqrt{b}} = \sqrt{\frac{a}{b}}$.

Example 1

Calculate

a. $\frac{\sqrt{2}}{\sqrt{3}}$

b. $\frac{-\sqrt{18}}{\sqrt{2}}$

Solutions:

a. $\frac{\sqrt{2}}{\sqrt{3}} = \sqrt{\frac{2}{3}}$

b. $\frac{-\sqrt{18}}{\sqrt{2}} = -\sqrt{\frac{18}{2}} = -\sqrt{9} = -3$

When $a > 0, b > 0$, then $a\sqrt{b} = \sqrt{a^2b}$, $\sqrt{a^2b} = a\sqrt{b}$.

Example 2

Convert each of the following in $\sqrt{a}$ form.

a. $3\sqrt{2}$

b. $5\sqrt{5}$

Solutions:

a. $3\sqrt{2} = \sqrt{3^2 \times 2} = \sqrt{18}$

b. $5\sqrt{5} = \sqrt{5^2 \times 5} = \sqrt{125}$

Exercise 2.15

1. Calculate.

a. $\frac{\sqrt{27}}{\sqrt{3}}$ b. $\frac{\sqrt{12}}{\sqrt{3}}$ c. $\frac{\sqrt{14}}{\sqrt{7}}$ d. $\frac{-\sqrt{15}}{\sqrt{3}}$

2. Convert each of the following in $\sqrt{a}$ form.

a. $4\sqrt{2}$ b. $5\sqrt{3}$ c. $-3\sqrt{7}$ d. $7\sqrt{6}$

Activity 2.13

Give an example which satisfies:

a. Two neither repeating nor terminating decimals whose sum is rational.

b. Two neither repeating nor terminating decimals whose sum is irrational.

c. Any two irrational numbers whose difference is rational.

d. Any two irrational numbers whose difference is irrational.

Example 1

Simplify each of the following.

a. 0.131331333 ... + 0.535335333 ...

b. 0.4747747774 ... − 0.252552555 ...

Solutions:

a. 0.131331333 ... + 0.535335333 ... = 0.666 ...

b. 0.4747747774 ... − 0.252552555 ... . = 0.222 ...

Example 2

Simplify each of the following.

a. $2\sqrt{3} + 4\sqrt{3}$ **b.** $3\sqrt{5} - 2\sqrt{5}$

Solutions:

a. $2\sqrt{3} + 4\sqrt{3} = (2 + 4)\sqrt{3} = 6\sqrt{3}$

b. $3\sqrt{5} - 2\sqrt{5} = (3 - 2)\sqrt{5} = \sqrt{5}$

Example 3

Simplify each of the following.

a. $\sqrt{8} + \sqrt{2}$ **b.** $\sqrt{12} - 5\sqrt{3}$

c. $\sqrt{18} + \sqrt{50}$ **d.** $\sqrt{72} - \sqrt{8}$

Solutions:

First, convert $\sqrt{a^2 b}$ into $a\sqrt{b}$ form.

a. $\sqrt{8} + \sqrt{2} = \sqrt{4 \times 2} + \sqrt{2} = \sqrt{2^2 \times 2} + \sqrt{2} = 2\sqrt{2} + \sqrt{2} = 3\sqrt{2}$

b. $\sqrt{12} - 5\sqrt{3} = \sqrt{2^2 \times 3} - 5\sqrt{3} = 2\sqrt{3} - 5\sqrt{3} = -3\sqrt{3}$

c. $\sqrt{18} + \sqrt{50} = \sqrt{3^2 \times 2} + \sqrt{5^2 \times 2} = 3\sqrt{2} + 5\sqrt{2} = 8\sqrt{2}$

d. $\sqrt{72} - \sqrt{8} = \sqrt{2^2 \times 3^2 \times 2} - \sqrt{2^2 \times 2}$

$$= \sqrt{(2 \times 3)^2 \times 2} - \sqrt{2^2 \times 2} = 6\sqrt{2} - 2\sqrt{2} = 4\sqrt{2}$$

Exercise 2.16

Simplify each of the following.

a. 0.56456445644456 ... − 0.111 ...

b. 2.1010010001 ... + 1.0101101110 ...

c. $2\sqrt{5} - 4\sqrt{5}$

d. $\sqrt{18} + 2\sqrt{2}$

e. $\sqrt{3} + 4\sqrt{3}$

f. $\sqrt{80} - \sqrt{20}$

g. $5\sqrt{8} + 6\sqrt{32}$

h. $\sqrt{8} + \sqrt{72}$

i. $\sqrt{12} - \sqrt{48} + \sqrt{\frac{3}{4}}$

j. $\sqrt{5} - \sqrt{45}$

2.4 Real Numbers

In the previous two sections, you have learnt about rational numbers and irrational numbers. Rational numbers are either terminating or repeating. You can locate these numbers on the number line. You have also discussed that it is possible to locate irrational numbers which are neither repeating nor terminating on the number line.

Activity 2.14

1. Can you think of a set which consists both rational numbers and irrational numbers?
2. What can you say about the correspondence between the points on the number line with rational and irrational numbers?

Based on the reply for the above questions and recalling the previous lessons, you can observe that every decimal number (rational or irrational) corresponds to a point on the number line. So that there should be a set which consists both rational and irrational numbers. This leads to the following definition.

Definition 2.7 Real numbers

A number is called a real number if and only if it is either a rational number or an irrational number. The set of real numbers is denoted by $\mathbb{R}$, and is described as the union of the sets of rational and irrational numbers. We write this mathematically as

$$\mathbb{R} = \{x : x \text{ is a rational number or irrational number}\}.$$

The following diagram indicates the set of real number, $\mathbb{R}$, is the union of rational and irrational numbers

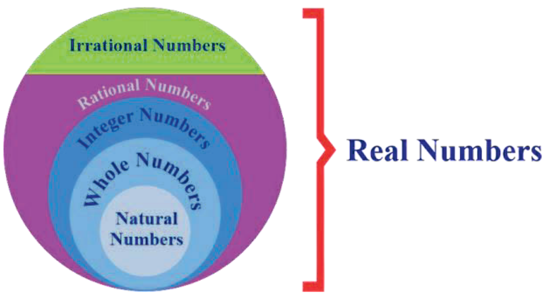


Figure 2.7: Real numbers

Comparing real numbers

You have seen that there is a one to one correspondence between a point on the number line and a real number.

- Suppose two real numbers a and b are given. Then one of the following is true $a < b$ or $a = b$ or $a > b$ (This is called trichotomy property).
- For any three real numbers a, b and c, if $a < b$ and $b < c$, then $a < c$ (called transitive property order)

Applying the above properties we have

- For any two non-negative real numbers a and b if $a^2 < b^2$, then $a < b$.

Example

Compare each pair (you can use Scientific calculator whenever it is necessary).

a. $-2,\ -8$ **b.** $\frac{5}{8}, 0.8$

c. $\frac{\sqrt{2}}{3},\ 0.34$ **d.** $\frac{\sqrt{3}}{6}, \frac{2}{3}$

Solution:

a. The location of these two numbers on the number line help us to compare the given numbers. Here, -8 is located at the left of -2 so that $-8 < -2$.

b. Find the decimal representation of $\frac{5}{8}$, that is $\frac{5}{8} = 0.625$.

Hence, $\frac{5}{8} < 0.8$.

c. Using a scientific calculator we approximate $\frac{\sqrt{2}}{3} \approx 0.4714$. Then, $\frac{\sqrt{2}}{3} > 0.34$.

d. Here use the third property given above. Square the two numbers

$$\left(\frac{\sqrt{3}}{6}\right)^2 = \frac{3}{36} \text{ and } \left(\frac{2}{3}\right)^2 = \frac{4}{9} = \frac{16}{36}.$$

Hence, it follows $\frac{\sqrt{3}}{6} < \frac{2}{3}$.

Exercise 2.17

Compare each of the following pairs (using $>$ or $<$, or $=$).

a. $-7, -3$

b. $\frac{2}{5}, 0.6$

c. $\frac{\sqrt{5}}{2}, 0.123$

d. $\frac{\sqrt{2}}{3}, \frac{4}{9}$

e. $0.236,\ 0.256$

f. $-7\sqrt{145}\ , \sqrt{7}$

g. $-0.135, -0.1351$

h. $6\sqrt{5}\ , 5\sqrt{7}$

i. $\frac{\sqrt{5}}{3}, 0.234$

j. $2 + \sqrt{3}, 4$

k. $\frac{22}{7}, \pi$

Determining real numbers between two numbers

Example

Find a real number between 2 and 4.2.

Solution:

Take the average of the two numbers, that is, $\frac{2+4.2}{2} = \frac{6.2}{2} = 3.1$.

This number is between 2 and 4.2. That is, $2 < 3.1 < 4.2$. Also take the average of 2 and 3.1. That is $\frac{2+3.1}{2} = 2.55$. Again take the average of 3.1 and 4.2, that is $\frac{3.1+4.2}{2} = 3.65$. These numbers 2.55 and 3.65 are also between 2 and 4.2.

Therefore, we conclude that there are infinitely many real numbers between two real numbers.

Exercise 2.18

1. Find at least two real numbers between:
   a. -0.24 and -0.246
   b. $\frac{3}{5}$ and $\frac{2}{7}$
   c. $\sqrt{2}$ and $\sqrt{3}$
2. How many real numbers are there between two real numbers?

2.4.1 Intervals

Activity 2.15

1. List some of the real numbers between 2 and 6 including 2.
2. List some of the real numbers between $\sqrt{2}$ and 3 excluding the two numbers.
3. Can you denote real numbers in (1) and (2) symbolically?

From activity 2.15 above, you observed that there should be a notation to describe such real numbers.

Case I (Representing real numbers between two points)

A real interval is a set which contains all real numbers between two numbers. We have the following.

Consider a real number x between two real numbers a and b such that $a \leq x \leq b$. Types of intervals are shown in the table 2.1 below.

Table 2.1

Inequality	Interval Notation	Graph on Number Line	Description
$a < x < b$	(a, b)		x strictly between a and b
$a \leq x < b$	$[a, b)$		x between a and b, including a
$a < x \leq b$	$(a, b]$		x between a and b, including b
$a \leq x \leq b$	$[a, b]$		x between a and b, including a and b

Case II (Representing real numbers in numbers with one end point)

Notation:

The symbol '∞' read as 'infinity' means endlessness or absence of end to the right and '$-\infty$' read as 'negative infinity' means endlessness or absence of end to the left.

Using a point a or b, ∞ and $-\infty$, the intervals contain real numbers as shown below in table 2.2.

Table 2.2

Inequality	Interval Notation	Graph on Number Line	Description
$x > a$	(a, ∞)		x is greater than a
$x < a$	$(-\infty, a)$		x is less than a
$x \geq a$	$[a, \infty)$		x is greater than or equal to a
$x \leq a$	$(-\infty, a]$		x is less than or equal to a

Example

Represent each of the following using interval notation.

a. Real numbers between 3 and 5 including both points.

b. Real numbers between -2 and $\frac{3}{2}$ including -2 and excluding $\frac{3}{2}$.

c. Real numbers on the left of 2 excluding 2.

d. Real numbers on the right of 0 including 0.

Solution:

a. The interval [3,5] represents a set which contains infinitely many real numbers between 3 and 5 including both end points. This is shown below on the number line in figure 2.7(a), where

- the filled circle ● on the number line is to mean <u>included</u> and the empty circle ○ is <u>not included</u> in the interval
- [ ] indicates the interval include the end points and ( ) indicates the interval do not include the end points.

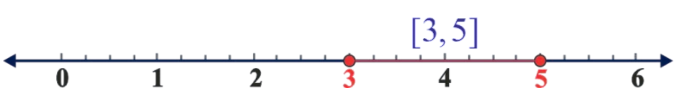


Figure 2.8(a)

b. An interval $[-2, \frac{3}{2})$ represent a set which contain infinitely many real numbers between -2 and $\frac{3}{2}$ including -2 in figure 2.8(b) below.

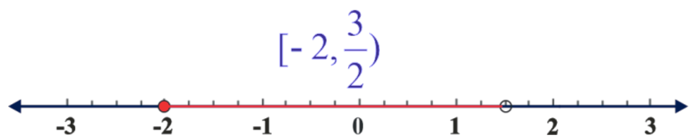


Figure 2.8(b)

c. The real numbers on the left of 2 excluding 2 is denoted by $(-\infty, 2)$ as shown in figure 2.8 (c) below.

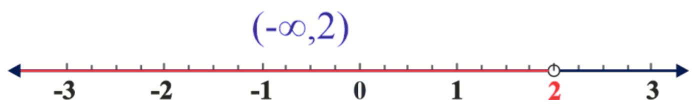


Figure 2.8(c)

d. The real numbers on the right of 0 including 0 is denoted by $[0, \infty)$ as shown in figure 2.8 (d) below.

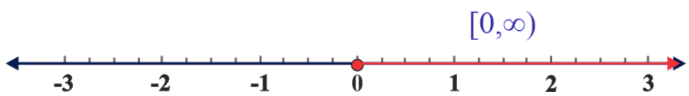


Figure 2.8(d)

Exercise 2.19

1. Represent each of the following using interval notations.
   a. Real numbers between -3 and 8 including both end points.
   b. Real numbers between 4 and 6 excluding the end points.
   c. Real numbers on the right of -1 including -1.
2. Represent each of the following intervals on the number line.
   a. $[1,3)$
   b. $(-\infty, -5]$
   c. $\left(\frac{1}{2}, 3\right)$
   d. $(-2,4) \cup [5, \infty)$

2.4.2 Absolute values

Definition 2.8

The absolute value of a real number, denoted by $|x|$, is defined as

$$|x| = \begin{cases} x \text{ if } x \geq 0, \\ -x \text{ if } x < 0 \end{cases}.$$

Example

Find the absolute value of each of the following real numbers.

a. 4 **b.** -2 **c.** $\frac{1}{2}$

d. -0.4 **e.** $-\sqrt{2}$ **f.** $2-\sqrt{8}$

Solution:

a. So using the definition,

$|4| = 4$, since 4 is a positive number and its absolute value is the number itself.

b. The given number is -2 which is less than zero. But its distance from zero is 2 units. Hence, $|-2| = 2$.

c. Following similar procedure as that of (a) and (b), $\left|\frac{1}{2}\right| = \frac{1}{2}$ (since $\frac{1}{2}$ is a positive number).

d. $|-0.4| = 0.4$ (its distance from zero is 0.4).

e. $|-\sqrt{2}| = \sqrt{2}$. (its distance from zero is $\sqrt{2}$).

f. $|2-\sqrt{8}| = \sqrt{8}-2$ ($\sqrt{8} = \sqrt{2^3} = 2\sqrt{2}$, also $\sqrt{2} \approx 1.414$ so that $2\sqrt{2} > 2$. Hence, $2-\sqrt{8} < 0$ and $\sqrt{8}-2 > 0$).

Activity 2.16

Consider two numbers $a = -3$ and $b = 4$, then

i. Find the number of units between a and b by counting on the number line.

ii. Find $|a - b|$

iii. Find $|b - a|$.

iv. What do you conclude from ii and iii.

You will learn more applications of absolute value in unit three.

Exercise 2.20

1. Find the absolute value of each of the following.

   a. 8 b. -5 c. $-\frac{2}{3}$ d. 0.7

   e. $\sqrt{3}$ f. $1 - \sqrt{2}$ g. $3 - \sqrt{5}$ h. $-0.\bar{1}2$

2. Find the distance between the given numbers on the number line.

   a. 2 and 10 b. 7 and -9 c. -49 and -100 d. -50 and 50

3. Determine the unknown 'x' for each.

   a. $|x| = 8$ b. $|x| = 0$ c. $|x| = -4$

   d. $|x| + 3 = 4$ e. $|x| - 3 = -2$

4. The coldest temperature on the Earth, -89°C, was recorded in 1983 at Vostok station, Antarctica. The hottest temperature on the Earth, 58°C, was recorded at Al'Aziziyah, Libya. Calculate the temperature range on the Earth.

2.4.3 Exponents and radicals

You have learnt about multiplication of two or more real numbers in lower grades. So that you can easily write

$$3 \times 3 \times 3 = 27.$$
$$4 \times 4 \times 4 = 64.$$
$$2 \times 2 \times 2 \times 2 \times 2 = 32.$$

Think of the situation when 14 is multiplied 20 times. It takes more time and space to write.

$$\underbrace{14 \times 14 \times 14 \times \ldots \times 14}_{20\ times}$$

This difficulty can be overcome by introduction of exponential notations. In this subsection, you shall learn the meaning of such notations, laws of exponents, radicals, simplification of expressions using the laws of exponents and radicals.

Definition 2.9

If a is a real number and n is a positive integer, then

$\underbrace{a \times a \times a \times a \times a \times \ldots \times a}_{n\ times} = a^n$ is an exponential expression, where a is the base and n is the exponent or power.

For instance: **a)** 2^3 means $2 \times 2 \times 2 = 8$

b) $(-3)^4$ means $(-3) \times (-3) \times (-3) \times (-3) = 81$

c) -2^3 means $-(2 \times 2 \times 2) = -8$.

Activity 2.17

1. Complete the following table.

Power form of a number	2^4	2^3	2^2	2^1	2^0	2^{-1}	2^{-2}	2^{-3}	2^{-4}
A number	16	8	4						

Hint:- To fill the incomplete boxes of the table, observe that the first row of the table is moving to the right by decreasing the power by 1. So that, the boxes on the second row will be determined by dividing the previous number by 2.

2. From the above pattern, find the relationship between 2^n and 2^{-n}.
3. What can you generalize about the relationship between a^n and a^{-n} for a real number $a \neq 0$ and positive integer n.

Note

1. Zero exponent: If $a \neq 0$, then $a^0 = 1$.
2. Negative exponent: If $a \neq 0$ and n is positive integer, then $a^{-n} = \frac{1}{a^n}$.

The above activity leads to the following definition.

Definition 2.10

If $a^2 = b$, then a is a square root of b. If $a^3 = b$, then a is a cube root of b. If n is a positive integer and $a^n = b$, then a is called the n^{th} root of b.

Definition 2.11 Principal n^{th} root

If b is any real number and n is a positive integer greater than 1, then the principal n^{th} root of b is denoted by $\sqrt[n]{b}$ is defined as

$$\sqrt[n]{b} = \begin{cases} \text{the positive } n^{th} \text{ root of } b\text{, if } b > 0, \\ \text{the negative } n^{th} \text{ root of } b\text{, } if\ b < 0 \text{ and } n \text{ is odd,} \\ 0\text{, if } b = 0. \end{cases}$$

Notations

- $\sqrt[n]{b}$ called radical expression where $\sqrt{\ }$ is radical sign, n is index and b is radicand. When the index is not written, the radical sign indicates the principal square root.

Definition 2.12 The $\left({}^{1}/_{n}\right)^{th}$ power

i) If $b \in \mathbb{R}$ and n is an odd positive integer greater than 1, then $b^{\frac{1}{n}} = \sqrt[n]{b}$.

ii) If $b \geq 0$ and n is an even positive integer greater than 1, then $b^{\frac{1}{n}} = \sqrt[n]{b}$.

Example 1

Find **a.** $\sqrt{9}$ **b.** $\sqrt{0.01}$

c. $\sqrt[4]{81}$ **d.** $\sqrt[5]{-100{,}000}$.

Solution:

a. 3

b. $\sqrt{0.01} = 0.1$ because $(0.1)^2 = 0.01$.

c. $\sqrt[4]{81} = 3$ because $3^4 = 81$.

d. $\sqrt[5]{-100{,}000} = -10$ because $(-10)^5 = -100{,}000$.

Example 2

Express each of the following in exponential form.

a. $\sqrt{3}$ **b.** $\sqrt[3]{6}$ **c.** $\frac{1}{\sqrt[4]{10}}$

Solution:

a. $\sqrt{3} = 3^{\frac{1}{2}}$ **b.** $\sqrt[3]{6} = 6^{\frac{1}{3}}$ **c.** $\frac{1}{\sqrt[4]{10}} = 10^{-\frac{1}{4}}$

Exercise 2.21

1. Find

   a. $\sqrt{36}$ b. $\sqrt{0.016}$ c. $\sqrt[4]{16}$ d. $\sqrt[3]{-1000}$.

2. Express each of the following in exponential form

   a. $\sqrt{5}$ b. $\sqrt[7]{7}$ c. $\frac{1}{\sqrt[3]{3^4}}$

   d. $\left(\sqrt[4]{81}\right)^2$ e. $\sqrt[3]{\frac{2}{5}}$

3. Simplify each of the following

   a. $(-27)^{\frac{1}{3}}$ b. $32^{\frac{1}{5}}$ c. $\frac{\sqrt[3]{125}}{\sqrt{625}}$

   d. $\sqrt{0.09}$ e. $\sqrt[4]{(-5)^4}$

Law of exponents

Activity 2.18

For the given table below,

i) Determine the simplified form of each term.

ii) Compare the value you obtained for (I) and (II) for each row.

No	I	II
1	$2^3 \times 3^3$	$(2 \times 3)^3$
2	$\frac{2^5}{2^3}$	2^{5-3}
3	$(3^2)^3$	$(3^3)^2$
4	$(3 \times 4)^2$	$3^2 \times 4^2$
5	$2^2 \times 2^3$	2^{2+3}

The above activity 2.17 leads you to have the following laws of exponents.

For any $a, b \in \mathbb{R}$ and $n, m \in \mathbb{N}$, the following holds:

- ✓ $a^n \times a^m = a^{n+m}$
- ✓ $\frac{a^n}{a^m} = a^{(n-m)}$, where $a \neq 0$
- ✓ $(a^n)^m = (a^m)^n = a^{nm}$
- ✓ $(a \times b)^n = a^n \times b^n$
- ✓ $\frac{a^n}{b^n} = \left(\frac{a}{b}\right)^n$, for $b \neq 0$.

Note that these rules also applied for $\left(\frac{1}{n}\right)^{th}$ power and rational power of the form $a^{\frac{m}{n}} = (a^{\frac{1}{n}})^m$.

Example 1

Use the above rules and simplify each of the following.

a. $7^{\frac{1}{2}} \times 7^{\frac{3}{2}}$ **b.** $(25 \times 9)^{\frac{1}{2}}$ **c.** $\frac{3^{\frac{5}{2}}}{3^{\frac{1}{2}}}$ **d.** $\left(\frac{8}{27}\right)^{\frac{1}{3}}$ **e.** $(81)^{\frac{1}{4}}$

Solution:

a. Using $a^n \times a^m = a^{n+m}$, $7^{\frac{1}{2}} \times 7^{\frac{3}{2}} = 7^{\left(\frac{1}{2}+\frac{3}{2}\right)} = 7^2 = 49.$

b. Using $(a \times b)^n = a^n \times b^n$,

$(25 \times 9)^{\frac{1}{2}} = 25^{\frac{1}{2}} \times 9^{\frac{1}{2}} = (5^2)^{\frac{1}{2}} \times (3^2)^{\frac{1}{2}} = 5 \times 3 = 15.$

c. Using $\frac{a^n}{a^m} = a^{(n-m)}$, $\frac{3^{\frac{5}{2}}}{3^{\frac{1}{2}}} = 3^{\left(\frac{5}{2}-\frac{1}{2}\right)} = 3^2 = 9.$

d. Using $\frac{a^n}{b^n} = \left(\frac{a}{b}\right)^n$, for $b \neq 0$, $\left(\frac{8}{27}\right)^{\frac{1}{3}} = \frac{8^{\frac{1}{3}}}{27^{\frac{1}{3}}} = \frac{(2^3)^{\frac{1}{3}}}{(3^3)^{\frac{1}{3}}} = \frac{2}{3}$.

e. Using $(a^n)^m = (a^m)^n = a^{nm}$, $(81)^{\frac{1}{4}} = (3^4)^{\frac{1}{4}} = 3.$

Example 2

Simplify each of the following.

a. $\sqrt{50}$ **b.** $\frac{32^{\frac{1}{4}}}{162^{\frac{1}{4}}}$ **c.** $\sqrt[3]{16} \times \sqrt[3]{4}$ **d.** $\left(8 + \sqrt[3]{27}\right)^{\frac{1}{3}}$

Solution:

a. To simplify this, we use different rules of radicals and exponent. 50 can be written as a product of a square number 25 and another number 2.

That is, $\sqrt{50} = \sqrt{25 \times 2} = \sqrt{25} \times \sqrt{2} = \sqrt{5^2} \times \sqrt{2} = 5\sqrt{2}$

b. Both numbers 32 and 162 are multiples of 2 so that we can express these numbers as a power of 2, that is $\frac{32^{\frac{1}{4}}}{162^{\frac{1}{4}}} = \frac{(2 \times 16)^{\frac{1}{4}}}{(2 \times 81)^{\frac{1}{4}}} = \frac{2^{\frac{1}{4}} \times 16^{\frac{1}{4}}}{2^{\frac{1}{4}} \times 81^{\frac{1}{4}}} = \frac{16^{\frac{1}{4}}}{81^{\frac{1}{4}}} = \frac{(2^4)^{\frac{1}{4}}}{(3^4)^{\frac{1}{4}}} = \frac{2}{3}$.

c. Express 16 as a product of 4, so that $\sqrt[3]{16} \times \sqrt[3]{4} = \sqrt[3]{16 \times 4} = \sqrt[3]{4 \times 4 \times 4} = (4^3)^{\frac{1}{3}} = 4.$

d. First express 27 as a multiple of 3 and apply the rule of exponents. This results,

$$\left(8+\sqrt[3]{27}\right)^{\frac{1}{3}}=\left(8+\sqrt[3]{3^3}\right)^{\frac{1}{3}}=\left(8+3^{\frac{3}{3}}\right)^{\frac{1}{3}}=(8+3)^{\frac{1}{3}}=11^{\frac{1}{3}}=\sqrt[3]{11}.$$

Exercise 2.22

1. Use the laws of exponents and simplify each of the following.

   a. $5^{\frac{1}{3}}\times 5^{\frac{2}{3}}$ b. $(16\times 49)^{\frac{1}{2}}$ c. $\frac{2^{\frac{10}{3}}}{2^{\frac{1}{3}}}$ d. $32^{\frac{2}{5}}$

2. Simplify each of the following.

   a. $\sqrt{125}$ b. $\frac{9^{\frac{1}{3}}}{243^{\frac{1}{3}}}$ c. $\sqrt[5]{0.00032}$

   d. $2^{\frac{3}{2}}\times\sqrt{2}$ e. $\frac{5\sqrt{24}\div 2\sqrt{50}}{3\sqrt{3}}$ f. $\sqrt[3]{0.001}+\sqrt[4]{0.0081}$

   g. $(5^{-1})^3\times 5^{\frac{5}{4}}\times\sqrt{25}\times\left(\sqrt[3]{125}\right)^3$ h. $\left(\sqrt{0.64}\right)\times\left(\sqrt[3]{\frac{1}{64}}\right)^2\times 32^{\frac{1}{5}}$

   i. $3^{\frac{1}{4}}\times 27^{\frac{1}{4}}$ j. $\sqrt{5+2\sqrt{6}}+\sqrt{8-2\sqrt{15}}$

Addition and subtraction of radicals

Activity 2.19

What do you say about these statements? Which one is correct?

a. $\sqrt{2}-\sqrt{2}=0$ b. $\sqrt{16}-\sqrt{4}=\sqrt{12}$

c. $\sqrt{2}+\sqrt{3}=\sqrt{5}$ d. $4\sqrt{2}-\sqrt{2}=3\sqrt{2}$

Definition 2.13

Radicals that have the same index and the same radicand are said to be like radicals.

For instance:- **i)** $2\sqrt{5}, -\frac{2}{3}\sqrt{5}$ and $\sqrt{5}$ are like radicals

ii) $\sqrt{2}$ and $\sqrt{3}$ are not like radicals

iii) $\sqrt{3}$ and $\sqrt[3]{3}$ are not like radicals

By following the same procedure of addition and subtraction of similar terms, we can add or subtract like radicals. In some cases we got like radicals after simplification.

Example

Simplify each of the following.

a. $\sqrt{3} + \sqrt{12}$ **b.** $2\sqrt{18} - \sqrt{2} + \sqrt{8}$

Solution:

a. $\sqrt{3} + \sqrt{12} = \sqrt{3} + \sqrt{4 \times 3} = \sqrt{3} + 2\sqrt{3} = (1 + 2)\sqrt{3} = 3\sqrt{3}.$

b. $2\sqrt{18} - \sqrt{2} + \sqrt{8} = 2\sqrt{9 \times 2} - \sqrt{2} + \sqrt{4 \times 2}$

$= 6\sqrt{2} - \sqrt{2} + 2\sqrt{2}$

$= (6 - 1 + 2)\sqrt{2}$ (taking common term)

$= 7\sqrt{2}$

Exercise 2.23

Simplify each of the following.

a. $\sqrt{12} - \sqrt{3}$ b. $\sqrt[3]{54} - \sqrt[3]{2}$ c. $3\sqrt[3]{27} - \sqrt[3]{125} + \sqrt{169}$

Operations on real numbers

You discussed the rules of exponents, operations on radicals and how to simplify numbers related to radicals. Recall that a real number system is a collection of rational and irrational numbers. You studied operations on rational numbers on section 2.2 and operations on irrational numbers on section 2.3.2. We can summarize operations on rational and irrational numbers as follows:

- The rational numbers satisfy the commutative, associative and distributive laws for addition and multiplication. Moreover, if we add, subtract, multiply or divide (except by zero) two rational numbers, we still get a rational number (that is, rational numbers are 'closed' with respect to addition, subtraction, multiplication and division).
- Irrational numbers also satisfy the commutative, associative and distributive laws for addition and multiplication. However, the sum, difference, quotients and products of irrational numbers are not always irrational.

Activity 2.20

1. Identify whether each of the following is rational or irrational.

   a. $\pi + 2$ b. $\sqrt{2} - 1$ c. $(1 - \sqrt{3}) \times (1 + \sqrt{3})$ d. $\frac{\sqrt{5}}{2}$

   e. $\frac{22}{7} \times \frac{22}{7}$ f. $\sqrt{7}(4 - \sqrt{7})$ g. $0.2323\ldots + 0.3232\ldots$ h. $\frac{8}{\sqrt{2}}$

2. Give an example for each (if possible).

   a. Two irrational numbers where their product is rational.

   b. A rational and irrational numbers and their sum is rational.

   c. A rational and irrational numbers and their difference is rational.

   d. A rational and irrational numbers and their quotient is irrational.

Now the above activity leads to draw the following conclusions.

- The sum and difference of rational and irrational numbers is irrational.
- The product, quotient of non-zero rational number and an irrational number is an irrational number.

Let us see some examples of four operations on the set of real numbers.

Example 1

If $x = 5\sqrt{3} + 2\sqrt{2}$ and $y = \sqrt{2} - 4\sqrt{3}$, then find

a. the sum of x and y. **b.** the difference between y and x

Solution:

a. $x + y = 5\sqrt{3} + 2\sqrt{2} + \sqrt{2} - 4\sqrt{3}$

$= 5\sqrt{3} - 4\sqrt{3} + 2\sqrt{2} + \sqrt{2}$ (rearranging terms)

$= (5 - 4)\sqrt{3} + (2 + 1)\sqrt{2}$ (taking out common factor)

$= \sqrt{3} + 3\sqrt{2}$

b. $x - y = 5\sqrt{3} + 2\sqrt{2} - (\sqrt{2} - 4\sqrt{3})$

$= 5\sqrt{3} + 4\sqrt{3} + 2\sqrt{2} - \sqrt{2}$

$= (5 + 4)\sqrt{3} + (2 - 1)\sqrt{2}$

$= 9\sqrt{3} + \sqrt{2}$

Example 2

Find the product of

a. $3\sqrt{3}$ and $2\sqrt{2}$

b. $5\sqrt{2}$ and $\frac{\sqrt{2}}{5}$.

Solution:

a. $3\sqrt{3} \times 2\sqrt{2} = 6 \times \sqrt{3} \times \sqrt{2} = 6\sqrt{6}$

b. $5\sqrt{2} \times \frac{\sqrt{2}}{5} = \frac{5\times\sqrt{2}\times\sqrt{2}}{5} = 2$

Example 3

Divide

a. $6\sqrt{8}$ by $3\sqrt{2}$

b. $\sqrt{3}$ by $\left(2\sqrt{2} \times 3\sqrt{3}\right)$.

Solution:

a. $6\sqrt{8} \div 3\sqrt{2} = \frac{6\sqrt{8}}{3\sqrt{2}} = 2\sqrt{\frac{8}{2}} = 2 \times 2 = 4$

b. $\sqrt{3} \div \left(2\sqrt{2} \times 3\sqrt{3}\right) = \frac{\sqrt{3}}{(2\sqrt{2}\times3\sqrt{3})} = \frac{\sqrt{3}}{6\sqrt{6}} = \frac{1}{6} \times \sqrt{\frac{3}{6}} = \frac{1}{6\sqrt{2}}$

Since a real number is a union of rational and irrational numbers, it is closed under addition, subtraction, multiplication and division (excluding division by zero).

Exercise 2.24

1. If $x = 4\sqrt{2} + 7\sqrt{5}$ and $y = \sqrt{2} - 3\sqrt{5}$, then find
   a. the sum of x and y.
   b. the difference between y and x
2. Find the product of
   a. $2\sqrt{5}$ and $4\sqrt{3}$
   b. $3\sqrt{3}$ and $\frac{\sqrt{3}}{3}$
   c. $\sqrt{3} - \sqrt{2}$ and $3\sqrt{3} - 4\sqrt{2}$
3. Divide
   a. $4\sqrt{6}$ by $2\sqrt{2}$
   b. $10\sqrt{2}$ by $5\sqrt{18}$
   c. $\sqrt{5}$ by $(3\sqrt{2} \times 4\sqrt{5})$
4. Compute each of the following.
   a. $5\sqrt{2} + 2\sqrt{3} + 3\sqrt{3} - \sqrt{2}$
   b. $\sqrt{145} - \sqrt{232} + \sqrt{261}$

Activity 2.21

1. Determine the opposite of each of the following real numbers
   a. $\sqrt{3}$
   b. $-\pi$
   c. $0.\bar{6}1$
   d. $\frac{1}{2\sqrt{5}}$
   e. 0
   f. $\sqrt{3} - \sqrt{2}$
2. Determine the reciprocal of each of the following real numbers.
   a. $\frac{-\sqrt{3}}{2}$
   b. $3.1\bar{4}34$
   c. $3\sqrt{3} - 1$
   d. $\frac{\sqrt[3]{4}}{1-\sqrt{6}}$
   e. $3^{\frac{2}{3}}$

From the above examples and activities, we have the following basic properties with addition and multiplication of real numbers.

Closure property:

The set of real numbers $\mathbb{R}$ is closed under addition and multiplication. This means that the sum and product of any two real numbers is always a real number. In other words, for all $a, b \in \mathbb{R}$, $a + b \in \mathbb{R}$ and ab or $a \times b \in \mathbb{R}$.

Commutative property

Addition and multiplication are commutative in $\mathbb{R}$: That is, for all $a, b \in \mathbb{R}$,

i) $a + b = b + a$

ii) $ab = ba$

Associative property

Addition and multiplication are associative in $\mathbb{R}$: That is, for all $a, b, c \in \mathbb{R}$,

i) $(a + b) + c = a + (b + c)$

ii) $(ab)c = a(bc)$.

Existence of additive and multiplicative identities:

There are real numbers 0 and 1 such that:

i) $a + 0 = 0 + a = a$ for all $a \in \mathbb{R}$.

ii) $a \times 1 = 1 \times a = a$ for all $a \in \mathbb{R}$.

Existence of additive and multiplicative inverses:

i. For each $a \in \mathbb{R}$ there exists $-a \in \mathbb{R}$ such that $a + (-a) = 0 = (-a) + a$, and $-a$ is called the additive inverse of a.

ii. For each non-zero $a \in \mathbb{R}$, there exists $\frac{1}{a} \in \mathbb{R}$ such that $\frac{1}{a} \times a = 1 = a \times \frac{1}{a}$ and $\frac{1}{a}$ is called the multiplicative inverse or reciprocal of a.

Distributive property:

Multiplication is distributive over addition; that is, if $a, b, c \in \mathbb{R}$,

i) $a(b + c) = ab + ac$.

ii) $(b + c)a = ba + ca$.

Example

Find the additive and multiplicative inverse for each of the following:

a. $\sqrt{2}$ **b.** $\frac{3}{4}$ **c.** 0.5

Solution:

a. Additive inverse of $\sqrt{2}$ is $-\sqrt{2}$ and multiplicative inverse of $\sqrt{2}$ is $\frac{1}{\sqrt{2}}$.

b. Additive inverse of $\frac{3}{4}$ is $\frac{-3}{4}$ and multiplicative inverse of $\frac{3}{4}$ is $\frac{4}{3}$.

c. Additive inverse of 0.5 is -0.5 and multiplicative inverse of 0.5 is 2 (since $0.5 = \frac{1}{2}$).

Exercise 2.25

1. Find the additive and multiplicative inverse of each of the following.

   a. $\sqrt{3}$ b. $-\frac{2}{5}$

   c. 1.3 d. $0.\overline{1}$

2. Identify the property of real numbers that is applied in the following statements.

   a. $\sqrt{3}(\sqrt{3}-2) = 3 - 2\sqrt{3}$

   b. $5 + (-8 + 2) = (5 - 8) + 2$

   c. $6 \times 4 \times \frac{1}{6} = 6 \times \frac{1}{6} \times 4$

3. Does every real number have a multiplicative inverse? Explain.

4. You define a new mathematical operation using the symbol (*). This operation is defined as $a * b = 2a + b$.

   a. Is this operation commutative? Explain.

   b. Is this operation associative? Explain.

2.4.4 Limit of accuracy

Measurements play an important role in daily life because they are useful to do basic tasks, such as take a child's temperature with a thermometer, make time estimations, measure out medicine and find weights, areas and volumes of different materials or substances. In the process of measurement, exact value may not be obtained so that you will be forced to take approximate value.

In this subtopic you will learn certain mathematical concepts related to approximation like rounding of numbers, significant figures (s.f.), decimal place (d.p.) and accuracy.

Activity 2.22

1. Round off the number 45,676 to the nearest.

   a. 100 b. 1000

2. Write the number 8.426

   a. to one decimal place

   b. to two decimal places.

3. Write the number 28.79 to three significant figures.

Rounding

Rounding off is a type of estimation. Estimation is used in everyday life and also in subjects like Mathematics and Physics. Many physical quantities like the amount of money, distance covered, length measured, etc., are estimated by rounding off the actual number to the nearest possible whole number and for decimals at various places of hundreds, tens, tenths, etc.

Rule for rounding whole numbers

When rounding numbers, you need to know the term "rounding digit". When working with whole numbers and rounding to the closest 10, the rounding digit is the second number from the right or the tens place. When rounding to the nearest hundred, the third place from the right is the rounding digit or hundreds place. To perform rounding:-

First, determine what your rounding digit is and then look for the digit at the right side.

- If the digit is 0, 1, 2, 3, or 4, do not change the rounding digit. All digits that are in the right hand side of the requested rounding digit become zero.
- If the digit is 5, 6, 7, 8, or 9 the rounding digit rounds up by one number. All digits that are on the right hand side of the requested rounding digit become zero.

Example 1

38,721 people live in a town. Round this number to various level of accuracy.

a. Nearest 100

b. Nearest 1,000

c. Nearest 10,000.

Solution:

a. To the nearest 100 the number of people would be rounded up to 38,700, since the rounding digit is 7, 2 is the number at the right side of it and keep the rounding digit as it is and all digits at the right hand side of it become zero.

b. To the nearest 1,000 this number would be rounded up to 39,000. The fourth number from the right, that is 8 is the rounding digit. 7 is the number at the right of this rounding digit, so that the rounding digit rounds up by one number so that it would be 9. All digits at the right hand side of this rounding digit become zero.

c. Similarly to the nearest 10,000 this number would be rounded up to 40,000, since 8 is at the right of the rounding digit(3), the rounding digit rounds up by one number so that it would be 4. All digits at the right hand side of this rounding digit become zero.

In this type of situation, it is unlikely that the exact number would be reported. In other words, the result is less accurate but easier to use.

Decimal place

A number can also be approximated to a given number of decimal places (d.p.). This refers to the number of figures written after a decimal point.

Example 2

a. Write 10.673 to 1 d.p.

b. Write 3.173 to 2 d.p.

Solution:

a. The answer needs to be written with one number after the decimal point. However, to do this, the second number after the decimal point also needs to be considered. If it is 5 or more, then the first number is rounded up.

That is,10.673 is written as 10.7 to 1 d.p.

b. The answer here is to be given with two numbers after the decimal point. In this case, the third number after the decimal point needs to be considered. As the third number after the decimal point is less than 5, the second number is not rounded up. That is, 3.173 is written as 3.17 to 2 d.p.

Note

To approximate a number to 1 decimal place means to approximate the number to the nearest tenth. Similarly approximating a number to 2 decimal places means to approximate to the nearest hundredth.

Significant figures

Numbers can also be approximated to a given number of significant figures (s.f.). In the number 43.25, the 4 is the most significant figure as it has a value of 40. In contrast, the 5 is the least significant as it only has a value of 5 hundredths. When we desire to use significant figures to indicate the accuracy of approximation, we count the number of digits in the number from left to right, beginning at the first non-zero digit. This is known as the number of significant figures.

Example 3

a. Write 2.364 to 2 s.f. **b.** Write 0.0062 to 1 s.f. **c.** Write 0.3041 to 2 s.f.

Solution:

a. We want to write only the two most significant digits. The third digit (6) needs to be considered to see whether the second digit (3) is to be rounded up or not. That is, 2.364 is written as 2.4 to 2 s.f.

b. Notice that in this case 6 and 2 are the only significant digits. The number 6 is the most significant digit and we will consider 2 to be rounded up or not. That is,0.0062 is written as 0.006 to 1 s.f. (since 2 is less than 5 we ignore its effect).

c. Here all digits after the decimal point are significant. So moving two steps from a decimal point to the right we will consider 4 to be rounded up or not. That is , 0.3041 is written as 0.30 to 2 s.f. (since 4 is less than 5 we ignore its effect).

Exercise 2.26

1. Round 86,343 to various level of accuracy
   a. Nearest 100 b. Nearest 1,000 c. Nearest 10,000.
2. Round each of the following to the nearest whole number.
   a. 35.946 b. 45.1999 c. $\frac{7}{8}$ d. $\sqrt{5}$
3. Express the following decimals to 1 d.p. and 2 d.p.
   a. 1.936 b. 4.752 c. 12.998
4. Write each of the following to the number of significant figures indicated in brackets.
   a. 28,645 (1 s.f.) b. 41,909 (3 s.f.) c. 4.5568 (3 s.f.)

Accuracy

In this lesson you will learn how to approximate upper and lower bounds for data to a specified accuracy (for example, numbers rounded off or numbers expressed to a given number of significant figures).

Activity 2.23

1. Round each of the following to 1 d.p.
   3.51, 3.48, 3.53, 3.4999, 3.49, 3.45, 3.47, 3.42, 3.57, 3.41, 3.59, 3.54.
2. Collect the numbers in (1) above which gives 3.5 after rounding and determine the maximum and minimum value from the list.

The above activity leads us to define upper and lower bound of a number.

Definition 2.14 Lower and upper bound

The upper and lower bounds are the maximum and minimum values that a number could have been before it was rounded.

Consider numbers 1.5, 1.50 and 1.500. They may appear to represent the same number but they actually do not. This is because they are written to different degrees of accuracy. 1.5 is rounded to one decimal place (or to the nearest tenths) and therefore any number from 1.45 up to but not including 1.55 would be rounded to 1.5. On the number line this would be represented as

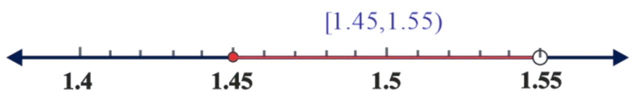


Figure 2.9

As an inequality, it would be expressed as $1.45 \leq 1.5 < 1.55$.

Here, 1.45 is known as the lower bound of 1.5, while 1.55 is known as the upper bound.

In order to find the lower and upper bounds of a rounded number, follow the following steps:

1) Identify the place value of the degree of accuracy stated.
2) Divide this place value by 2.
3) Add this amount (value in step2) to the given value to obtain the upper bound and subtract this value (value in step2) from the given value to obtain the lower bound.

Example 1

The weight of the tree is 872 kilograms to the nearest kilogram.

a. Find the upper and lower bounds where the weight of the tree lies.

b. If the tree's weight is wt kilograms, describe this range as inequality.

Solution:

a. 872 kg is rounded to the nearest kilogram,

Step 1. The place value is 1.

Step 2. Divide 1 by 2 and we get 0.5.

Step 3. Lower bound is $872 - 0.5 = 871.5$ and the upper bound is

$872 + 0.5 = 872.5$.

Therefore, the lower bound is 871.5 kg and the upper bound is 872.5.

b. When the tree's weight is kg, using the answer we got in (a), it could be expressed using inequality as $871.5 \leq wt < 872.5$.

Example 2

A number was given as 10.7 to 1 d.p. Find the lower and upper bounds of a number.

Solution:

Step 1. Place value of the degree of accuracy is $\frac{1}{10}$ or 0.1.

Step 2. Dividing 0.1 by 2 results 0.05.

Step 3. The lower bound is $10.7 - 0.05 = 10.65$ and upper bound is

$10.7 + 0.05 = 10.75$.

Exercise 2.27

1. The speed of a car is given as 45 m/s to the nearest integer.
   a. Find the lower and upper bounds within which the car speed can lie.
   b. If the car's speed is v m/s, express this range as inequality.
2. Find the lower and upper bounds of :
   a. 45 rounded to the nearest integer.
   b. 12.6 rounded to 1 d.p.
   c. 4.23 rounded to 2 d.p.
3. Express each of the following in between the upper and lower bounds.
   a. $x = 34.7$ b. $y = 21.36$ c. $z = 154.134$

Effect of operation on accuracy

Activity 2.24

1. Given $a = 5$ and $b = 2$, then write:-
   i. lower bound of a + lower bound of b
   ii. lower bound of a + upper bound of b
   iii. upper bound of a + lower bound of b
   iv. upper bound of a + upper bound of b

   Which of these gives the lowest value? Is this lowest value the sum of the lowest numbers?

   Which of these gives the highest value? Is this highest value the sum of the highest numbers?

2. Given $a = 9$ and $b = 6$, then write:-
   i. lower bound of a − lower bound of b
   ii. lower bound of a − upper bound of b
   iii. upper bound of a − lower bound of b
   iv. upper bound of a − upper bound of b

   Which of these gives the lowest value? Is this lowest value the difference of the lowest numbers?

   Which of these gives the highest value? Is this highest value the difference of the highest numbers?

When approximated numbers are added, subtracted and multiplied, their sum, difference and product give a range of possible answers.

Example

Given that a rectangular farmland has length 12 meters and width 6 meters.

a. Find the upper and lower bounds of the sum of length and width of the farmland.

b. Find the range of the area of the farmland (upper and lower bounds of the area).

Solution:

a. Let l be the length and w be the width of the farmland. That is, $l = 12$ meters and $w = 6$ meters. Then, $11.5 \leq l < 12.5$ and $5.5 \leq w < 6.5$.

The lower bound of the sum is obtained by adding the two lower bounds.

Therefore, the minimum sum is $11.5 + 5.5 = 17.0$

The upper bound of the sum is obtained by adding the two upper bounds.

Therefore, the maximum sum is $12.5 + 6.5 = 19.0$.

So, the sum lies between 17.0 meters and 19.0 meters.

b. The area of a rectangle A is determined by $A = l \times w$. So, let us determine the upper and lower bounds of the product of l and w. The lower bound of the product is obtained by multiplying the two lower bounds. Therefore, the minimum product is $11.5 \times 5.5 = 63.25$.

The upper bound of the product is obtained by multiplying the two upper bounds. Therefore, the maximum product is $12.5 \times 6.5 = 81.25$.

So, the product lies between 63.25 and 81.25. Hence the area of the farmland ranges between $63.25\ \text{m}^2$ and $81.25\ \text{m}^2$.

Exercise 2.28

1. Consider two numbers, 15 and 7.
   a. Find the upper and lower bounds of the sum of the two numbers.
   b. Find the range of the product of the two numbers.
2. Calculate the upper and lower bounds for the following calculations if each of the numbers is given to 1 decimal place.

   a. $5.4 + 6.2$ b. 4.6×2.7 c. $14.3 - 5.7$ d. $\frac{10.8}{3.1}$
3. Calculate upper and lower bounds for the area of a school football field shown below if its dimensions are correct to 1 decimal place.

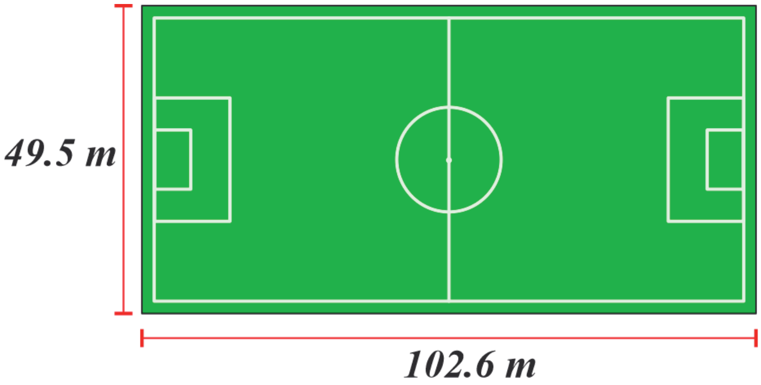


Figure 2.10

2.4.5 Standard notation (Scientific notation)

In science and technology, it is usual to see very large and very small numbers. For instance:

- Abay River travels 1,450,000 meters through Ethiopia and Sudan.
- The average distance from the earth to the moon is 382,500 kilometers.
- The mass of the hydrogen atom is 0.00000000000000000000000000167 kilograms.

Very large numbers and very small numbers may sometimes be difficult to work with or write. Hence you often write very large or very small numbers in scientific notation, also called standard notation.

Activity 2.25

Express each of the following as a multiple of 10.

i) 486.00017 ii) 14580,000,000,000

iii) 0.0006504 iv) 0.0000000000078436

As you have seen from the above activity, you might give different answer for the same question. This will lead us to have the following definition.

Definition 2.15

A number is said to be in scientific notation (or standard notation) if it is written as a product of the form $a \times 10^n$ where $1 \leq a < 10$ and n is an integer.

To convert numbers into scientific notation, we follow the following steps.

1. Move decimal point until there is one (non-zero) number in front of the decimal point.
2. The exponent of the power of 10 is determined by the number of places you moved the decimal point.
   a. If the original number is large, you moved the decimal point to the left and the number

      of places you moved the decimal point is exponent of the power of 10.
   b. If the original number is small, you moved the decimal point to the right, and the

      exponent of the power of 10 is negative integer of the number of places you moved the decimal point.
3. Put the number in the correct pattern for scientific notation.

Example 1

Express each of the following numbers into scientific notation

a. 1,856,700,000 **b.** 0.000000314

Solution:

9 8 7 6 5 4 3 2 1

a. **1856700000.**

1. First let us find a number between 1 and 10, that is a number which has 1 digit in front of the decimal point from the given number. The number is 1.8567.

2. The decimal point has been moved 9 places to the left so the exponent of the power of 10 is 9.
3. Here, the decimal points will move to the left and we will count the number to find n. That is, 1.8567×10^9.

1 2 3 4 5 6 7

b. 0.000000314

1. First let us find a number between 1 and 10, that is a number which has 1 digit in front of the decimal point from the given number. The number is 3.14.
2. The decimal point has been moved 7 places to the right so the exponent of the power of 10 is -7.
3. Here, the decimal points will move to the right and we will count the number to find n. That is, 3.14×10^{-7}.

Example 2

Express each of the following in decimal notation.

a. 5.37×10^6 **b.** 1.7×10^{-7}

Solution:

a. To change the given number in to ordinary decimal, the decimal point moves to the right 6 units. That is, $5.37 \times 1000000 = 5{,}370{,}000$.

b. Similarly, $1.7 \times 10^{-7} = 0.00000017$.

Exercise 2.29

1. Express each of the following in standard notation.

   a. 426,000 b. 158.762 c. 0.000089 d. 56,897.00547

2. Write each of the following in ordinary decimal notation

   a. 1.34×10^6 b. 3.3×10^{-3} c. $\frac{2}{5} \times 10^{-7}$

3. Find the simplified expression in standard notation form.

a. $(4.2 \times 10^3) + (1.6 \times 10^3)$

b. $(2.1 \times 10^3)(1.3 \times 10^4)$

c. $(1.5 \times 10^{-3})(3.1 \times 10^3)$

d. $\frac{(5.0\times10^5)}{(2\times10^{-2})}$

2.4.6 Rationalization

Whenever we have a ratio of numbers where the denominator is irrational, determining the quotient might be difficult. So, there is an intention of changing the denominator as rational.

For instance: $\frac{1}{\sqrt{3}}$ has an irrational denominator, $\sqrt{3}$. Here, our aim is changing the number $\frac{1}{\sqrt{3}}$ to an equivalent number with a rational denominator. What shall we do?

The technique of transferring the radical expression from the denominator to the numerator is called rationalizing the denominator (changing the denominator into a rational number).

Do you recall the existence of multiplicative identity 1 for the set of real numbers?

The number that can be used as a multiplier to rationalize the denominator is called the rationalizing factor which is equivalent to 1.

For example: If you have an irrational number $\sqrt{a}$ and if you need to rationalize the denominator of $\frac{1}{\sqrt{a}}$, the rationalizing factor will be $\frac{\sqrt{a}}{\sqrt{a}} = 1$.

Example

Rationalize the denominator for each of the following.

a. $\frac{2}{\sqrt{3}}$ **b.** $\frac{\sqrt{2}}{\sqrt{5}}$ **c.** $\frac{1}{\sqrt[3]{2}}$ (to change in to 2 as its denominator)

Solution:

a. The rationalizing factor for this number is $\frac{\sqrt{3}}{\sqrt{3}}$. So that $\frac{2}{\sqrt{3}} = \frac{2}{\sqrt{3}} \times \frac{\sqrt{3}}{\sqrt{3}} = \frac{2\sqrt{3}}{3}$. (why?)

b. The rationalizing factor is $\frac{\sqrt{5}}{\sqrt{5}}$. So, $\frac{\sqrt{2}}{\sqrt{5}} \times \frac{\sqrt{5}}{\sqrt{5}} = \frac{\sqrt{10}}{5}$. (why?)

c. We can take $\left(\sqrt[3]{2^2}/\sqrt[3]{2^2}\right)$ as rationalizing factor, $\frac{1}{\sqrt[3]{2}} = \frac{\sqrt[3]{2^2}}{\sqrt[3]{2} \times \sqrt[3]{2^2}} = \frac{\sqrt[3]{2^2}}{2} = \frac{\sqrt[3]{4}}{2}$.

Exercise 2.30

Rationalize the denominator of each of the following and write in simplest form.

a. $\frac{3}{\sqrt{5}}$ b. $\frac{2\sqrt{2}}{\sqrt{7}}$ c. $5\sqrt{\frac{1}{3}}$

More on rationalizations of denominators

Activity 2.26

1. What is the product of $\left(\sqrt{a} - \sqrt{b}\right)$ and $\left(\sqrt{a} + \sqrt{b}\right)$?
2. What is the product of $\left(a + \sqrt{b}\right)$ and $\left(a - \sqrt{b}\right)$?
3. What is the product of $\left(\sqrt{a} - b\right)$ and $\left(\sqrt{a} + b\right)$?

You might have got a rational number as a product for the above expressions.

This will lead you to have the following conclusion.

Table 2.3

No	Given number	Rationalizing factor
1	$\frac{1}{\sqrt{a} - \sqrt{b}}$	$\frac{\sqrt{a} + \sqrt{b}}{\sqrt{a} + \sqrt{b}}$
2	$\frac{1}{a + \sqrt{b}}$	$\frac{a - \sqrt{b}}{a - \sqrt{b}}$
3	$\frac{1}{\sqrt{a} - b}$	$\frac{\sqrt{a} + b}{\sqrt{a} + b}$
4	$\frac{1}{\sqrt{a} + \sqrt{b}}$	$\frac{\sqrt{a} - \sqrt{b}}{\sqrt{a} - \sqrt{b}}$

Example

Rationalize the denominator of

a. $\frac{1}{\sqrt{2}+1}$ **b.** $\frac{3}{\sqrt{5}-\sqrt{3}}$ **c.** $\frac{1}{\sqrt{3}-\sqrt{2}+1}$.

Solution:

a. The rationalizing factor is $\frac{\sqrt{2}-1}{\sqrt{2}-1}$, so that $\frac{1}{\sqrt{2}+1} = \frac{1}{\sqrt{2}-1} \times \frac{\sqrt{2}-1}{\sqrt{2}-1} = \frac{\sqrt{2}-1}{2-1} = \sqrt{2}-1$.

b. The rationalizing factor is $\frac{\sqrt{5}+\sqrt{3}}{\sqrt{5}+\sqrt{3}}$, so that $\frac{3}{\sqrt{5}-\sqrt{3}} = \frac{3}{\sqrt{5}-\sqrt{3}} \times \frac{\sqrt{5}+\sqrt{3}}{\sqrt{5}+\sqrt{3}} = \frac{3(\sqrt{5}+\sqrt{3})}{2}$.

c. Consider a rationalizing factor $\frac{\sqrt{3}-\sqrt{2}-1}{\sqrt{3}-\sqrt{2}-1}$. Now multiply both the numerator and denominator by this rationalizing factor.

$\frac{1}{\sqrt{3}-\sqrt{2}+1} \times \frac{\sqrt{3}-\sqrt{2}-1}{\sqrt{3}-\sqrt{2}-1} = \frac{\sqrt{3}-\sqrt{2}-1}{2(2-\sqrt{6})}$ (using multiplication and collecting like terms).

The denominator is still not rational. Hence, again we will take $\frac{2+\sqrt{6}}{2+\sqrt{6}}$ as a rationalizing factor. Multiplying both the numerator and the denominator by $\frac{2+\sqrt{6}}{2+\sqrt{6}}$, we obtain $\frac{\sqrt{3}-\sqrt{2}-1}{2\,(2-\sqrt{6})} \times \frac{(2+\sqrt{6})}{(2+\sqrt{6})} = \frac{\sqrt{2}-\sqrt{6}-2}{-4}$.

Exercise 2.31

Rationalize each of the following numbers.

a. $\frac{3}{2+\sqrt{5}}$ b. $\frac{1+\sqrt{2}}{\sqrt{3}-1}$ c. $\frac{2}{\sqrt{3}-\sqrt{2}}$ d. $\frac{2}{\sqrt{2}+\sqrt{3}+1}$

2.5 Applications

In the last 4 subsections, you have discussed number systems. In this subsection you will learn some application problems based on the covered topics.

Example 1

A small town has 530 flower pots. The gardener wants to arrange them in rows such that each row contains 21 flower pots. Find the number of completed rows and how many flower pots are left over.

Solution:

Total number of flower pots is 530. Number of flower pots in each row is 21.

Now, we have to find how many groups of 21 in 530. So, we use Euclid's Division Lemma, that is dividing 530 by 21 as shown in the right.

$$\begin{array}{r|l} & 25 \\ 21 & 530 \\ & \underline{42} \\ & 110 \\ & \underline{105} \\ & 5 \end{array}$$

The result indicates, the gardener can arrange the flower pots in 25 rows with each row consisting of 21 pots. Also the remaining number of flower pots is 5.

Example 2

The teacher wants to paste a square pieces of equal size colored papers on a white board measuring 20 cm by 50 cm. If only squares of length with natural number be considered, and the board is to be completely covered, find the largest possible length of the side of each square pieces.

Solution:

The dimension of the white board is 50 cm by 20 cm. There are different alternatives to construct the square pieces, in figure 2.11(a), we have 1 cm by 1 cm square pieces.

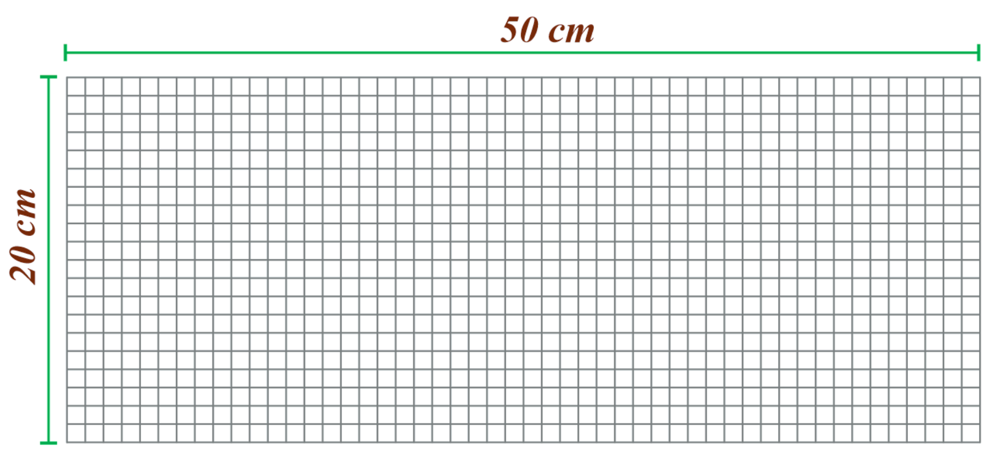


Figure 2.11 (a)

The following figure 2.11(b) is a 5 cm by 5 cm square pieces.

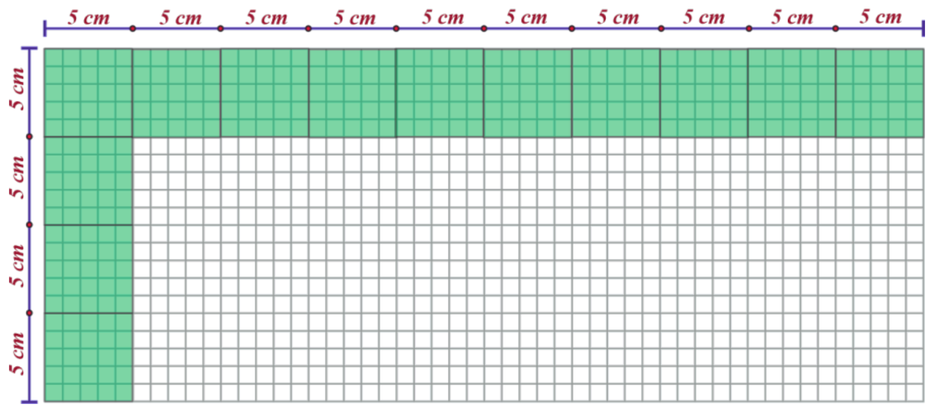


Figure 2. 11 (b)

And figure 2.11 (c) below, we have 10 cm by 10 cm.

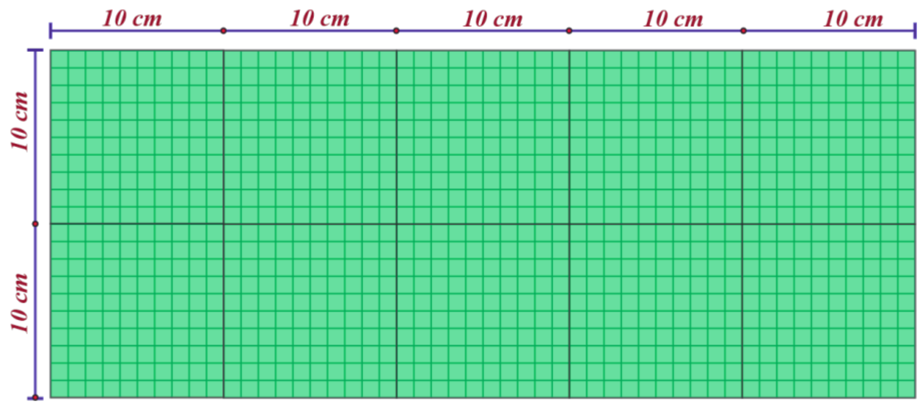


Figure 2.11 (c)

There are also other options to construct such square pieces with different lengths. Since we need only natural number length, we will try to find the GCF of 50 and 20. Using prime factorization $50 = 2 \times 5^2$ and $20 = 2^2 \times 5$. So, $\text{GCF}(50,20) = 10$. Hence, the largest possible length of the side of each square piece is 10 cm.

Exercise 2.32

1. Assume three strings of different lengths 78 cm, 117 cm and 351 cm cut into equal lengths. Find the greatest possible length of each piece.
2. Two 9th grade students A and B start running around the sport field together. Student A completes one round in 6 minutes while B takes 14 minutes to

complete the same round. After how many minutes will they meet again at the starting point for the second time?

3. There are 340 notebooks. A teacher is thinking to distribute them to 24 students equally as much as possible. Find number of notebooks each student gets and how many notebooks are left.

Summary

1. The set of Natural numbers, Integers and Rational numbers are denoted by $\mathbb{N}, \mathbb{Z}$ and $\mathbb{Q}$, respectively and described as

   $\mathbb{N} = \{1, 2, 3, \ldots\}$,

   $\mathbb{Z} = \{\ \ldots, -3, -2, -1,\ 0,\ 1,\ 2,\ 3,\ \ldots\}$ and

   $\mathbb{Q} = \left\{\frac{a}{b} : \text{a, b} \in \mathbb{Z} \text{ and } b \neq 0\right\}$.

2. Euclid's Division algorithm

   Given non-negative integer a and a positive integer b, there exist unique non-negative integers q and r satisfying:

   $a = (b \times q) + r$ with $0 \leq r < b$.

   From the above equation, a is called the dividend, q is called the quotient, b is called the divisor, and r is called the remainder.

3. a. A natural number that has exactly two distinct factors, namely 1 and itself is called a **prime number.**

   b. A natural number that has more than two factors is called a **composite number.**

   c. Prime factorization is a process of expressing a natural number as a product of prime numbers.

   d. **Fundamental theorem of arithmetic** states that every composite number can be expressed (factorized) as a product of primes, and this factorization is unique, apart from the order in which the prime factors occur.

4. a. Given two or more natural numbers, a number which is a factor of all of them is called a **common factor**.

a. The Greatest Common Factor (GCF) or Highest Common Factor (HCF) of two or more natural numbers is the greatest natural number of the common factors.

b. Two or more natural numbers where their GCF is 1 are called relatively prime.

c. For any two natural numbers a and b, the Least Common Multiple of a and b denoted by $\text{LCM}(a, b)$, is the smallest multiple of both a and b.

d. For any two natural numbers a and b, $\text{LCM}(a, b) \times \text{GCF}(a, b) = a \times b$.

5. **a.** Any rational number can be expressed as decimal by dividing the numerator a by the denominator b.

a. To change a rational number $^a/_b$ in to decimal form, one of the following cases will occur.

- The division process ends when a remainder of zero is obtained. Here, the decimal is called a terminating decimal.
- The division process does not terminate but repeats as the remainder never becomes zero. In this case the decimal is called a repeating decimal.

c. A repeating decimal also can be converted in to fractions.

6. Any terminating decimal or repeating decimal is a rational number where as a decimal number that is neither terminating nor repeating is an irrational number.

7. The sum of an irrational and a rational number is always an irrational number.

8. The set of irrational numbers is not closed with respect to addition, subtraction, multiplication and division.

9. A number is called a real number, if and only if it is either a rational number or an irrational number, that is $\mathbb{R} = \{x : x \text{ is rational or } x \text{ is irrational}\}$.

10. The absolute value of a real number x, denoted by $|x|$, is defined as

$$|x| = \begin{cases} x \text{ if } x \geq 0, \\ -x \text{ if } x < 0. \end{cases}$$

11. For any real number b and positive integer > 1, $b^{\frac{1}{n}} = \sqrt[n]{b}$ (whenever $\sqrt[n]{b}$ is a real number).

12. For all real numbers a and $b \neq 0$ for which the radicals are defined and for all integers $n \geq 2$.

i. $\sqrt[n]{a}\sqrt[n]{b} = \sqrt[n]{ab}$ **ii.** $\sqrt[n]{\frac{a}{b}} = \frac{\sqrt[n]{a}}{\sqrt[n]{b}}$

13. A number is said to be in scientific notation (or standard form) if it is written as the form $a \times 10^n$ where $1 \leq a < 10$ for an integer n.

Review Exercise

1. Identify whether each of the following numbers is divisible by $2, 3, 4, 5, 6, 8, 9$ or 10.

   a. 657 b. 10,222 c. 64,916

2. A four digit number has digit form '$a4b2$'. If it is divisible by 3, what are the possible values of $a + b$?

3. Fill in the blank space.

   a. ________is the sum of the smallest prime number and the smallest composite number.

   b. ________ is the smallest composite number.

4. Write each as prime factorization form.

   a. 57 b. 168 c. 536

5. Find the GCF and LCM of the numbers given below.

   a. 36 , 60 b. 84,224 c. 15,39,105 d. 16,20,48

6. The GCF of two numbers is 9 and the LCM of these two numbers is 54. If one of the numbers is 27, what is the other number?

7. Express each of the following rational numbers as decimals.

   a. $-2\frac{1}{5}$ b. $\frac{11}{7}$ c. $\frac{17}{33}$

8. Express each of the following decimals as a fraction in simplest form.

   a. 0.38 b. $2.\bar{1}2$ c. $-0.\bar{1}23$

9. Prove that there exists a real number between any two real numbers.

10. If $\frac{a}{b}$ and $\frac{c}{d}$ are two rational numbers, show that $\frac{a}{b} \times \frac{c}{d}$ is also rational number.

11. Find the absolute value of each of the following.

    a. $-10.1\bar{4}18$ b. $\sqrt[3]{6} - 2$ c. $\frac{a}{b}$, where $a, b \in \mathbb{R}$ and $a \geq 0$ and $b < 0$.

12. Give equivalent expression, containing fractional exponents, for each of the following

a. $\sqrt{14}$ **b.** $\sqrt{x+y}$ **c.** $\sqrt[5]{7}$ **d.** $\sqrt[3]{\frac{5}{6}}$

13. Describe the following numbers as fractions with rational denominators.

a. $-\frac{2}{\sqrt{5}}$ **b.** $\frac{4}{3-\sqrt{2}}$ **c.** $\frac{\sqrt{2}}{\sqrt{3}+\sqrt{2}}$ **d.** $\frac{1}{\sqrt{2}+\sqrt{5}+1}$

14. Find an irrational number between $\frac{1}{7}$ and $\frac{2}{7}$.

15. Given $\frac{\sqrt{2}+\sqrt{3}}{4+\sqrt{3}} = a\sqrt{2}+b\sqrt{3}+c\sqrt{6}+d$. Then, find a, b, c and d.

16. Simplify each of the following.

a. $7\sqrt{2}+\sqrt{8}-\sqrt{18}$ **b.** $\frac{1}{3}\times(\sqrt{5}-\sqrt{2})\times(\sqrt{5}+\sqrt{2})$ **c.** $3\sqrt{28}\div 8\sqrt{7}$

d. $\sqrt[4]{b^2\times\sqrt[3]{b^2}}$, for $b\geq 0$ **e.** $\frac{\sqrt{72}-3\sqrt{24}}{\sqrt{2}}$ **f.** $\frac{2\sqrt{72}}{3}-\frac{3\sqrt{128}}{4}+5\sqrt{\frac{1}{2}}$

17. Each of the following numbers is correct to one decimal place.

- Give the upper and lower bounds of each.
- Using x as the number, express the range in which the number lies as an inequality.

a. 2.4 **b.** 10.6 **c.** 2.0 **d.** -0.5

18. Express each of the following numbers in scientific notation.

a. 567,200,000 **b.** 0.00000774 **c.** 154.6×10^5

19. On the number line which one of the following real numbers is found farthest from the origin?

A. $\sqrt{1.44}$ **B.** $-\sqrt{3}$ **C.** $\frac{\sqrt{5}}{7}$ **D.** $-\sqrt{0.0001}$

20. According to the division algorithm, what should be the value of q and r respectively so that $736 = 12q + r$?

A. 60 and -4 **B.** 54 and 8 **C.** 60 and 4 **D.** 61 and 4

21. Suppose x is a rational number and y is an irrational number, then which one of the following is necessarily true?

A. xy is rational.

B. $\frac{x}{y}$ is irrational number.

C. $\frac{y}{x}$ is real number.

D. $x - y$ is irrational for any x.

22. Which one of the following statements is true about the set of real numbers $\mathbb{R}$?

A. Every real number has a multiplicative inverse.

B. Subtraction is associative over $\mathbb{R}$.

C. $-a$ is the additive inverse of a for every $a \epsilon \mathbb{R}$.

D. Division is commutative over $\mathbb{R}$.

23. When the positive integers a, b and c are divided by 13, the respective remainders are $9, 7$ and 10, respectively. Show that $a + b + c$ is divisible by 13.

24. There are 14 girls and 21 boys in a school. These students want to give a voluntary traffic safety service. Their school director assigned a task to be done in a team. Each team must have an equal number of boys and girls on it. What is the greatest number of teams the director can make if every student should be in a team? How many boys and girls will be in each team?

25. Floor of an ICT room of your school is to be fitted with a square marble of the largest possible size. The size of the room is 10 m by 7 m. What should be the size of the tiles required that has to be cut and how many such tiles are required?

UNIT 3

SOLVING EQUATIONS

Unit Outcomes

By the end of this unit, you will be able to:

- **Revise linear equation in one variable.**
- **Use different techniques of solving systems of equations in two variables.**
- **Solve simple equations involving absolute values.**
- **Compute equations involving exponents and radicals.**
- **Solve quadratic equations.**
- **Apply equations on real life situations.**

Unit Contents

KEY Words

- discriminant
- elimination method
- absolute value
- exponents
- completing the square
- graphical method
- linear equation
- factorization
- quadratic equation
- quadratic formula
- radicals
- substitution method

INTRODUCTION

Equations are like a balance weighing machine or scale. If you have seen a weighing machine, you might have thought that an equal amount of weight has to be placed on either side for the scale to be considered "balanced". If you add some weight to one side, the scale will point on one side and the two sides are no more in balance. Equations abide by the same reasonable judgment. Whatever is on one side of the equal sign must have exactly the same value on the other side; otherwise, it becomes an inequality.

In the earlier grades, you learned about linear equations in one variable and the methods to solve them. In the present unit, you will further discuss systems of linear equations in two variables, equations involving non-linear equations such as absolute values, exponents and radicals and quadratic equations.

3.1 Revision on linear equation in one variable

Activity 3.1

1. Determine each of the following is a linear equation in one variable or not.

a. $10x = 20$ b. $22x - 11 = 0$ c. $4x + 9 = -11$

d. $7x = 0$ e. $x^2 + x = 2$ f. $x + 2y = 1$

g. $4y + 5 = -18$ h. $2a + 3 = 4$ i. $3(n + 5) = 2(-6 - n) - 2n$

j. $3x + 6 = 0$ k. $y = x$ l. $x^2 - y = 0$

Activity 3.1 leads you to revise the rules used to solve linear equations that you learned in the previous grades.

Definition 3.1

When an equation in one variable has exponent equal to 1, it is said to be a linear equation in one variable. It is of the form $ax + b = 0$, where x is the variable, and a and b are real coefficients and $a \neq 0$. This equation has only one solution.

For solving a linear equation in one variable, the following steps are followed:

Step 1: Use LCM (Least Common Multiple) to clear the fractions if any.

Step 2: Simplify both sides of the equation.

Step 3: Isolate the variable.

Step 4: Verify your answer.

Example 1

Solve for x, if

a. $x + 2 = 0$ **b.** $3x + 4 = 10$

Solution:

a. Add -2 on both sides to isolate the variable x.

$$x + 2 + (-2) = 0 + (-2)$$

$$x = -2.$$

b. Similarly, $3x + 4 - 4 = 10 - 4$

$$3x = 6$$

$$x = \frac{6}{3} = 2. \text{ Hence, } x = 2$$

Example 2

Solve $\frac{5}{4}x + \frac{1}{2} = 2x - \frac{1}{2}$

Solution:

$\frac{5}{4}x + \frac{1}{2} = 2x - \frac{1}{2}$

$4\left(\frac{5}{4}x + \frac{1}{2}\right) = 4\left(2x - \frac{1}{2}\right)$ ....Using LCM to clear the fraction

$5x + 2 = 8x - 2$

$5x + 2 - 8x = 8x - 2 - 8x$

$-3x + 2 = -2$

$-3x + 2 + -2 = -2 + -2$

$-3x = -4$ ....After simplifying both sides of the equation

$x = \frac{4}{3}$ ....Isolating the variable

(Left hand side: LHS) = $\frac{5}{4}\left(\frac{4}{3}\right) + \frac{1}{2} = \frac{5}{3} + \frac{1}{2} = \frac{10+3}{6} = \frac{13}{6}$

(Right hand side: RHS) = $2\left(\frac{4}{3}\right) - \frac{1}{2} = \frac{8}{3} - \frac{1}{2} = \frac{16-3}{6} = \frac{13}{6}$

Thus, (LHS) = (RHS).............. Verifying the answer

Example 3

The length of the leg of an isosceles triangle is 6 cm more than its base length. If the perimeter of the triangle is 54 cm, then find the lengths of the sides of the triangle.

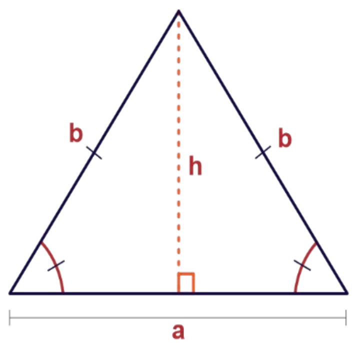


Figure 3.1

Solution:

Let us assume the base measure a cm.

Hence, each of the legs measures $b = (a + 6)\ cm$

The perimeter of a triangle is the sum of the legs of the three sides. The equations are formed and solved as follows:

$$a + 2(a + 6) = 54$$

$$a + 2a + 12 = 54$$

$$3a = 42$$

$$a = 14$$

Then, $b = 14 + 6$

$$b = 20$$

Hence, the length of the base is 14 cm and that of the two legs is 20 cm.

Exercise 3.1

1. Solve the following linear equations:
   a. $x + 7 = 0$
   b. $x + 4 = 9$
   c. $2x - 7 = 3$
   d. $3x - 7 = -10$
   e. $5x - 8 = 2x - 2$
   f. $10(x + 10) - 7 = 13$
   g. $7(y - 2) + 21 = (2y)(3)$
2. Verify whether $x = -3$ is a solution of the linear equation $10x + 7 = 13 - 5x$ or not.
3. If the sum of two consecutive numbers is 67, then find the numbers.

3.2 Systems of linear equations in two variables

Activity 3.2

Two pencils and one eraser cost Birr 5 and three pencils and two erasers cost Birr 8. Let the price of a pencil and an eraser be x and y Birr, respectively. Express the statements with x and y.

A system of linear equations consists of two or more linear equations made up of two or more variables such that all equations in the system are considered simultaneously.

To find the unique solution to a system of linear equations, we must find a numerical value for each variable in the system that will satisfy all equations in the system at the same time. Some linear systems may not have a solution and others may have an infinite number of solutions. In order for a linear system to have a unique solution, there must be at least as many equations as there are variables. This even does not guarantee a unique solution.

In this section, we will discuss systems of linear equations in two variables, which consist of two equations that contain two different variables.

To solve linear equations with two variables, there are three methods. These are solving using tables, solving by substitution method and elimination method.

Definition 3.2

An equation of the type $ax + by = c$ where a, b and c are arbitrary constants and $a \neq 0$, $b \neq 0$, is called a linear equation in two variables.

3.2.1 Solving systems of linear equations in two variables using tables

Example 1

$\begin{cases} 2x + y = 15 \\ 3x - y = 5 \end{cases}$ is a system of linear equations in two variables. Find the solution.

Solution:

Solve each equation for y. Then, find the pair of x and y that satisfies each equation as follows:

$y = -2x + 15$

x	0	1	2	3	**4**	5	6
y	15	13	11	9	**7**	5	3

$y = 3x - 5$

x	0	1	2	3	**4**	5	6
y	-5	-2	1	4	**7**	10	13

Hence, the pair of x and y that satisfies both equation is $(4, 7)$. This is the solution. The solution to a system of linear equations in two variables is any ordered pair that satisfies each equation independently. In this example, after filling the tables above, the ordered pair $(4, 7)$ is the solution to the system of linear equations.

Example 2

Check whether the ordered pair $(5,1)$ is a solution to the following linear system of equations or not.

$$\begin{cases} x + 3y = 8 \\ 2x - y = 9 \end{cases}$$

Solution:

Substitute the ordered pair $(5,1)$ into both equations.

First equation: $(LHS) = 5 + 3(1) = 8$, $(RHS) = 8$,

Thus, $(LHS) = (RHS)$, True

Second equation: $(LHS) = 2(5) - 1 = 9$, $(RHS) = 9$

Thus, $(LHS) = (RHS)$, True

The ordered pair $(5,1)$ satisfies both equations, so it is the solution to the system.

Exercise 3.2

1. Solve the following linear equation in two variables using table.

$$\begin{cases} 2x - y = -1 \\ x + y = 4 \end{cases}$$

2. Determine whether the ordered pair $(8,5)$ is a solution to the following linear equation in two variables or not.

$$\begin{cases} 5x - 4y = 20 \\ 2x - 3y = -1 \end{cases}$$

3.2.2 Solving systems of linear equations in two variables by substitution

One of the methods of solving a system of linear equations is substitution method. The method involves solving one of the equations in terms of the other variable and then substitute the result into the other equation to solve the second variable.

Steps in solving systems of linear equations in two variables by substitution

Step 1. Solve one of the two equations for one of the variables in terms of the other.

Step 2. Substitute the expression for this variable into the second equation, then solve for the remaining variable.

Step 3. Substitute that solution into either of the original equations to find the value of the first variable. If possible, write the solution as an ordered pair.

Step 4. Check the solution in both equations.

Example 1

Solve the following system of linear equations by substitution method.

$$\begin{cases} -x + y = -5 \\ 2x - 5y = 1 \end{cases}$$

Solution:

First, solve the first equation for y (or solve the second equation for y)

$y = x - 5$ (Substitute this into the second equation).

$$2x - 5(x - 5) = 1$$

$$-3x + 25 = 1$$

$$-3x = -24$$

$$x = 8$$

Now, we substitute $x = 8$ into the first equation or second equation and solve for y.

$$-8 + y = -5$$

$$y = -5 + 8 = 3.$$

Therefore, the solution of the above system of linear equations is $(x, y) = (8, 3)$.

Example 2

Solve the following system of linear equations using substitution method.

$$\begin{cases} x - 6y = -3 \\ 3x - 5y = 4 \end{cases}$$

Solution:

1st equation: $x = 6y - 3$. Substitute this into the 2nd equation.

$$3(6y - 3) - 5y = 4$$

$$18y - 9 - 5y = 4$$

$$13y = 13$$

$$y = 1$$

Then, $x = 6y - 3 = 6(1) - 3 = 3$

Exercise 3.3

1. Solve the following system of linear equations using substitution method.

a. $\begin{cases} 4x + y = 23 \\ y = 6x + 3 \end{cases}$

b. $\begin{cases} x = y - 6 \\ 5x + 2y = -2 \end{cases}$

c. $\begin{cases} x + 5y = 72 \\ x = 7y \end{cases}$

d. $\begin{cases} 2x - 7y = 2 \\ 3x + y = -20 \end{cases}$

e. $\begin{cases} 4x + y = 23 \\ y = 6x + 3 \end{cases}$

f. $\begin{cases} x = y - 6 \\ 5x + 2y = -2 \end{cases}$

g. $\begin{cases} 2x + 3y = -13 \\ 4x - 10 = 6y \end{cases}$

h. $\begin{cases} x + 3y = 22 \\ x = 4 \end{cases}$

i. $\begin{cases} \frac{x}{3} + \frac{y}{6} = 3 \\ \frac{x}{2} - \frac{y}{4} = 1 \end{cases}$

j. $\begin{cases} 8x + 3y = 9 \\ -\frac{x}{6} + \frac{y}{2} = 2 \end{cases}$

3.2.3 Solving systems of linear equations in two variables by Addition (Elimination) method

Another method of solving systems of linear equations is the addition method, also called the elimination method. In this method, we add two terms with the same variable, but opposite coefficients, so that the sum is zero. Of course, not all systems are set up with the two terms of one variable having opposite coefficients. Often, we must adjust by multiplying one or both of the equations so that one of the variables will be eliminated by addition.

The following are steps to solve system of equations using the addition (elimination) method.

Step 1. Write both equations with x and y variables on the left side of the equal sign and constants on the right.

Step 2. Write one equation above the other, lining up corresponding variables. If one of the variables in the top equation has the opposite coefficient of the same variable in the bottom equation, add the equations together, eliminating one variable. If not, use multiplication by a nonzero number so that one of the variables in the top equation has the opposite coefficient of the same variable in the bottom equation, and then add the equations to eliminate the variable.

Step 3. Solve the resulting equation for the remaining variable.

Step 4. Substitute that value into one of the original equations and solve for the second variable.

Step 5. Check the solution by substituting the values into the other equation.

Example 1

Solve the following using addition (elimination) method.

$$\begin{cases} x + 2y = 1 \\ -x + y = 5 \end{cases}$$

Solution:

Both equations are already set equal to a constant. Notice that the coefficient of x in the second equation, -1, is the opposite of the coefficient of x in the first equation, 1. We can add the two equations to eliminate x without needing to multiply by a constant.

$$\begin{cases} x + 2y = 1 \\ -x + y = 5 \end{cases}$$

$$0 + 3y = 6$$

$$y = 2$$

Now, we have eliminated x, and we have solved for y. Then, we substitute this value for y into one of the original equations and solve for x as follows:

$$-x + 2 = 5$$

$$x = -3$$

Therefore, the solution is: $x = -3, y = 2$

Example 2

Solve the following using addition (elimination) method.

$$\begin{cases} 4x + 2y = 28 \\ x + 2y = 13 \end{cases}$$

Solution:

Note that that the coefficient of y in the first and the second equation are the same.

$$\begin{cases} 4x + 2y = 28 \\ -(x + 2y = 13) \end{cases}$$

$$3x = 15$$

$$x = 5$$

Substitute in the second equation of the original, we have

$$x + 2y = 13$$

$$5 + 2y = 13$$

$$2y = 8$$

$$y = 4$$

Therefore, the solution is: $x = 5, y = 4$

Exercise 3.4

Solve the following system of linear equations using addition (elimination) method.

a. $\begin{cases} x + 3y = 5 \\ -x + y = -1 \end{cases}$

b. $\begin{cases} 2x + y = 12 \\ 2x - y = 12 \end{cases}$

c. $\begin{cases} 2x + y = 20 \\ x - y = 4 \end{cases}$

d. $\begin{cases} 2x + 5y = 12 \\ 2x + y = 4 \end{cases}$

Addition (Elimination) method 2

Example 1

Solve the following using substitution (elimination) method.

$$\begin{cases} 3x + 4y = 2 \\ x - 2y = 4 \end{cases}$$

Solution:

Adding these equations as presented will not eliminate a variable. However, we see that the first equation has $3x$ in it and the second equation has x. So, if we multiply the second equation by -3, the x-terms will add to zero.

$$\begin{cases} 3x + 4y = 2 \\ -3(x - 2y) = -3(4) \end{cases}$$

$$\begin{cases} 3x + 4y = 2 \\ -3x + 6y = -12 \end{cases}$$

$$10y = -10$$

$$y = -1$$

Substituting in the second equation of the original, we have

$$x - 2y = 4$$

$$x - 2(-1) = 4$$

$$x = 4 - 2 = 2$$

Example 2

Solve the following using substitution (elimination) method.

$$\begin{cases} 2x + 3y = -2 \\ 5x + 2y = 17 \end{cases}$$

Solution:

Multiply the first equation and second equation by 2 and by -3, respectively.

$$\begin{cases} 2(2x + 3y) = 2(-2) \\ -3(5x + 2y) = -3(17) \end{cases}$$

$$\begin{cases} 4x + 6y = -4 \\ -15x - 6y = -51 \end{cases}$$

$$-11x \quad = -55$$

$$x = 5$$

Substitute in the first equation of the original, we have

$$2(5) + 3y = -2$$

$$3y = -12$$

$$y = -4$$

Therefore, the solution is $x = 5, \ y = -4$.

Exercise 3.5

Solve the following using Addition (elimination) method.

a. $\begin{cases} 2x - 3y = 5 \\ x - 2y = 6 \end{cases}$

b. $\begin{cases} 2x - 7y = 2 \\ 3x + y = -20 \end{cases}$

c. $\begin{cases} 2x + 3y = 5 \\ 3x + 5y = 7 \end{cases}$

d. $\begin{cases} 9x + 3y = 38 \\ 5x + y = 22 \end{cases}$

e. $\begin{cases} -3x + y = -9 \\ -7x + 4y = 6 \end{cases}$

Word Problems involving equations in two variables

Example

Three pens and two books cost Birr 118 and one pen and two books cost Birr 106. Find the cost of one pen and one book, separately.

Solution:

If p is the cost of one pen and b is the cost of one book, then $\begin{cases} 3p + 2b = 118 \\ p + 2b = 106 \end{cases}$

Solve the second equation for p in terms of b.

$p = -2b + 106$. Substitute this into the first equation, $3(-2b + 106) + 2b = 118$.

Hence, $b = 50$ and $p = 6$.

Exercise 3.6

1. Two pens and three books cost Birr 170.00 and five pens and one book cost Birr 100.00. Find the cost of one pen and one book, separately.
2. One bread and one injera cost Birr 15 and ten breads and four injera cost Birr 90. Find the cost of a bread and one injera, separately.
3. A jacket and two Ethiopian dresses cost Birr 4000.00 and five jackets and three Ethiopian dresses cost Birr 9500.00. Find the cost of one jacket and one Ethiopian dress, separately.
4. If twice the age of a son is added to age of a father, then the sum is 56. If twice the age of the father is added to the age of son, then the sum is 82. Find the ages of the father and the son.
5. In a two-digit number, the sum of the digits is 13. Twice the tens digit exceeds the units digit by two. Find the numbers.
6. I am thinking of a two-digit number. If I write 3 to the left of my number, and double this three digit number, the result is 27 times my original number. What is my original number?

3.2.4 Graphical method of solving system of linear equations in two variables

Graphical method is one of the methods of solving systems of linear equations. Solving a linear system in two variables by graphing works well when the solution consists of integer values, but if our solution contains decimals or fractions, it is not the most precise method. For a system of linear equations in two variables, we can determine both the type of system and the solution by graphing the system of equations on the same set of axes.

To solve systems of linear equations using graphs, use the following steps:

Step 1. Graph the first equation.

Step 2. Graph the second equation using the same coordinate system.

Step 3. Determine whether the lines intersect, are parallel, or are the same line.

Step 4. Identify the solution to the system. If the lines intersect, identify the point of intersection.

Example 1

Solve the following system of equations graphically.

$$\begin{cases} 2x + y = -8 \\ x = y - 1 \end{cases}$$

Solution: First**,** solve the first equation for y, that is

$$2x + y = -8$$
$$y = -2x - 8$$

Solve the second equation for y, that is,

$$x - y = -1$$
$$y = x + 1$$

Second, graph both equations on the same set of axes:

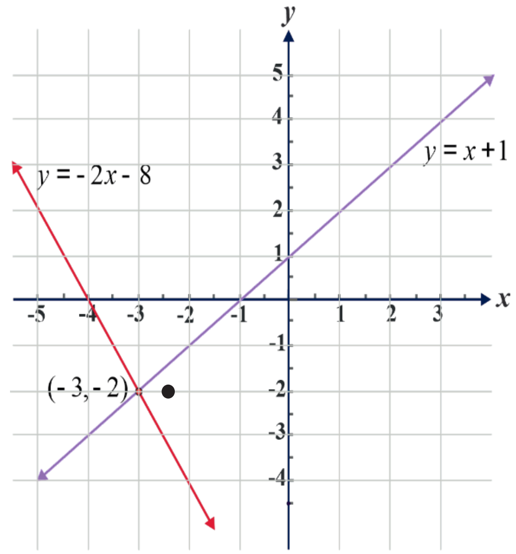


Figure 3.2

The graph above (Fig 3.2) indicates the intersection point of the lines: $y = -2x - 8$ and $y = x + 1$. The lines appear to intersect at the point$(-3, -2)$. This intersection point is the solution of the system. We can check to make sure that this is the solution to the system by substituting the ordered pair into both equations.

First equation: $(LHS) = 2(-3) + (-2) = -8$, $(RHS) = -8$,

Thus, $(LHS) = (RHS)$, True

Second equation: $(LHS) = -3$, $(RHS) = -2 - 1 = -3$

Thus, $(LHS) = (RHS)$, True

The solution to the system is the ordered pair $(-3, -2)$.

Example 2

Solve the following system of equations graphically.

$$\begin{cases} 2x - y = -1 \\ 4x - 2y = -8 \end{cases}$$

Solution:

First**,** solve the two equations for y.

First equation: $y = 2x + 1$

Second equation: $2y = 4x + 8$, $y = 2x + 4$

Then, draw the graphs of both equations. The graph on right (Fig 3.3) indicates that the two lines are parallel and do not intersect each other. This means that there is no solution to the system.

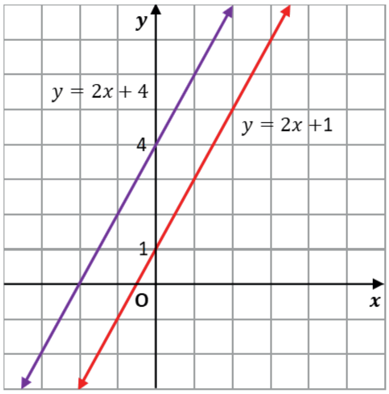


Figure 3.3

Example 3

Solve the following system of equations by graphical method.

$$\begin{cases} 3x + y = 6 \\ 6x + 2y = 12 \end{cases}$$

Solution:

When solving the two equations for y, we obtain $y = -3x + 6$

Graph the equation on the *xy*-coordinate plane.

In addition to considering the number of equations and variables, we can categorize systems of linear equations by the number of solutions.

Remark:

If a system of two equations has different slopes and intersects at one point in the plane, then the system is said to have exactly one solution. If the equations of the system have the same slope and the same y-intercepts, then the system has infinitely many solutions. In other words, the lines coincide so the equations represent the same line. Every point on the line represents a coordinate pair that satisfies the system. Thus, there are an infinite number of solutions.

Another type of system of linear equations is one in which the equations represent two parallel lines. The lines have the same slope and different y-intercepts. There are no points common to both lines; hence, there is no solution to the system. Below is a comparison of graphical representations of each type of system (fig 3.4).

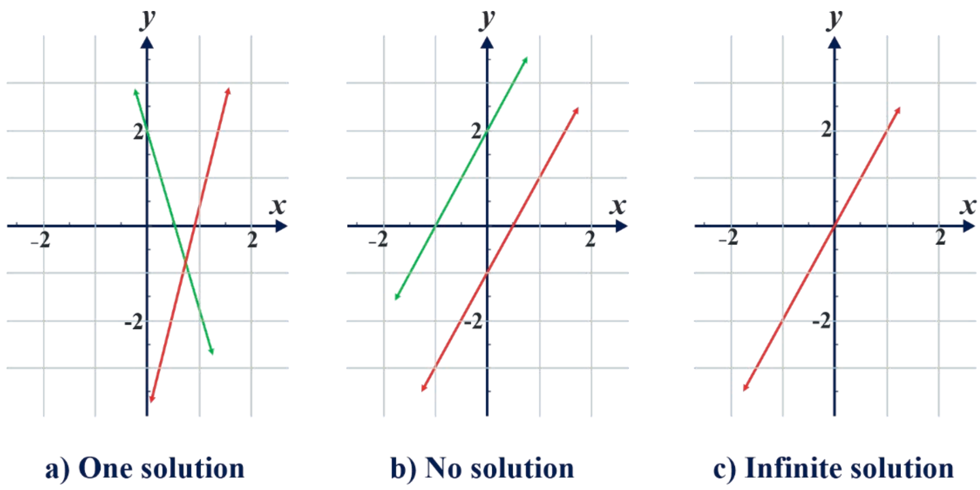


Figure 3.4

Exercise 3.7

1. Solve the following system of linear equations graphically.

   a. $\begin{cases} y = 2x + 1 \\ y = 4x + 3 \end{cases}$ b. $\begin{cases} x + y = 6 \\ -x + y = 2 \end{cases}$

2. Determine whether each of the following system of equations has one solution, infinite solution or no solution.

   a. $\begin{cases} 3x + y = 6 \\ 6x + 2y = 2 \end{cases}$ b. $\begin{cases} 3x + y = 6 \\ y = -3x + 6 \end{cases}$

   c. $\begin{cases} 2x - y = 6 \\ -2x + 3y = 2 \end{cases}$ d. $\begin{cases} 2x + 5y = 12 \\ x - \frac{5}{2}y = 4 \end{cases}$

3. What must be the value of k so that the system of linear equations

   $\begin{cases} x - y = 3 \\ 2x - 2y = k \end{cases}$ has

   i. infinitely many solution?

   ii. no solution?

   iii. unique solution?

4. Under what conditions on a and b will the following system of linear equation have no solution, one solution or infinitely many solutions?

$$\begin{cases} 2x - 3y = a \\ 4x - 6y = b \end{cases}$$

3.3 Solving nonlinear equations

3.3.1 Equations involving absolute value

Review of absolute value

In the previous sections, you worked with equations having variables x or y that can assume any value. But sometimes it becomes necessary to consider only non-negative values.

If you consider distance, it is always non-negative. The distance of a number x is located on the real line from the origin is either positive or zero. If x is at the origin distance is 0.

Activity 3.3

Compare each of the following numbers in absolute value.

a. 2 and -3

b. 6 and -5

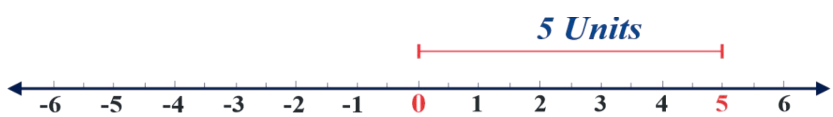
c. ¼ and -1/2

d. $-\sqrt{10}$ and 3

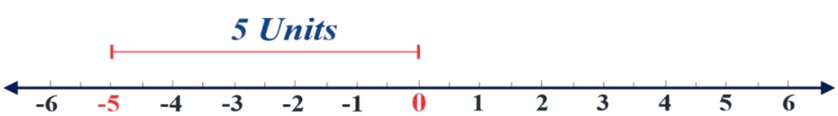
e. $-\frac{1}{3}$ and $-0.333...$

From unit two, recall that real numbers can be represented on a line as follows:

The absolute value of 5 is 5. The distance from 5 to 0 is 5 units.



The absolute value of –5 is 5. The distance from -5 to 0 is 5 units.



From this, it is possible to determine the distance of each point, representing a number, located far away from the origin or the point representing 0.

Example 1

Let A and B be points on a number line with coordinates 5 and −5, respectively. How far are the points A and B from the origin? How many points are there equal distances from the origin on a number line?

Solution:

The distance of A and B from the origin is 5 units long on the real line. There are infinitely many points that are equal distances from the origin on a number line.

Note

If Q is a point on a number line with coordinate a real number q, then the distance of Q from the origin is called the absolute value of q and is written as $|q|$.

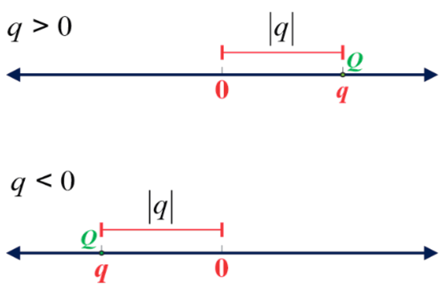


Figure 3.5

Example 2

The points represented by numbers 3 and –3 are located on the number line at an equal distance from the origin. Hence, $|3| = |-3| = 3$.

Example 3

Find the absolute value of each of the following real numbers.

a. -6 **b.** 7 **c.** -0.8

Solution:

a. $|-6| = 6$ **b.** $|7| = 7$ **c.** $|-0.8| = 0.8$

Exercise 3.8

Find the absolute value of the following:

a. $|-4|$

b. $|-\frac{1}{2}|$

c. $|0|$

Solving equations involving absolute value

Example 1

Solve the equation, $|x| = 4$

Solution:

$$|x| = 4$$
$$x = \pm 4$$

Properties of absolute value.

1. For any real number p, $|p| = |-p|$.

2. For any real number p, $|p| \geq 0$.

3. For any non- negative number p, $|x| = p$ means $x = p$ or $x = -p$.

Example 2

Solve the equation, $|x + 7| = 14$

Solution:

For any non- negative number p, $|x| = p$ means $x = p$ or $x = -p$.

So, you begin by making it into two separate equations and then solve them separately.

$|x + 7| = 14$ means:

$x + 7 = 14$ *or* $x + 7 = -14$

$x = 14 - 7$ *or* $x = -14 - 7$

$x = 7 \qquad or \qquad x = -21$

Hence, $x = -21,\ x = 7$ are solutions of the equation.

Remark: An absolute value equation has no solution if the absolute value expression equals a negative number since an absolute value can never be negative. For example, $|x| = -1$ has no solution. (Why?)

Example 3

Solve the absolute value equation, $2|2x - 1| - 1 = 5$

Solution:

$$2|2x - 1| = 6$$

$$|2x - 1| = 3$$

$$2x - 1 = 3 \; or \; -(2x - 1) = 3$$

$$x = 2 \; or \; x = -1$$

Hence, $x = -1,\ x = 2$ are solutions to the equation.

Exercise 3.9

1. Let A and B be points on a number line with coordinates 6 and -6, respectively. How far are the points A and B from the origin? How many points are there equal distances from the origin on a number line?
2. Solve the following absolute value equations.

a. $|x + 2| = 6$

b. $|5 - 3x| = 7$

c. $|x - 2| = 0$

d. $|x + 2| = -6$

e. $|2x - 1| + 3 = 6$

f. $|-2x + 7| = 25$

g. $|-8x - 2| - 6 = 10$

h. $-4|x + 3| = -28$

i. $\frac{|x+9|}{2} = 7$

j. $3|5 + 3x| + 1 = 7$

k. $4 - 6|4 + 8x| = -116$

3.3.2 Quadratic equations

Activity 3.4

Multiply the left side of each of the following. What do you get? What is the difference between these equations and linear equations?

1. $(x+3)(x-2)=0$
2. $(5x+1)(2x+4)=2$
3. $\left(\frac{1}{2}x-3\right)(x+5)=0$

Definition 3.3

An equation of the form $ax^2+bx+c=0$ where a, b, c, ∈ R and $a \neq 0$ is a quadratic equation. Here, a is called the leading coefficient, b is the middle term and c is the constant term.

Solving quadratic equations

There are three basic methods for solving quadratic equations: factorization (if possible), completing the square and the quadratic formula method.

Factorization method 1

Activity 3.5

Find two integers such that

a. the sum is 5 and the product is 6.
b. the sum is 1 and the product is -12.

How to solve a quadratic equation by factorization method?

1. Put all terms on one side of the equal sign, leaving zero on the other side.
2. Factorize the equation.

3. Set each factor equal to zero.
4. Solve each of these equations.
5. Check by inserting your answer in the original equation.

Example 1

Solve the quadratic equation $x^2 + 3x = 0$.

$$x(x + 3) = 0$$
$$x = 0 \ \ or \ x + 3 = 0$$
$$x = 0 \ \ or \ x = -3$$

Hence, the solution of the quadratic equation is $x = 0, \ x = -3$.

Example 2

Solve the quadratic equation: $x^2 - 6x - 16 = 0$

Solution:

Factorizing this (find two numbers whose sum is -6 and product is -16), we have -8 and 2. Hence, $(x - 8)(x + 2) = 0$

$$x - 8 = 0 \text{ or } x + 2 = 0$$

Recall that $ab = 0$ if and only if $a = 0$ or $b = 0$

$$x = 8, \ x = -2$$

Hence, the solution of the quadratic equation is $x = 8, \ x = -2$.

Exercise 3.10

Solve the following quadratic equations using factorization method.

a. $x^2 - 5x = 0$

b. $x^2 + 7x + 10 = 0$

c. $x^2 + x - 6 = 0$

d. $x^2 - 4x + 3 = 0$

Factorization Method 2

Example 1

Solve the quadratic equation $x^2 + 6x + 9 = 0$

Solution:

$(x+3)(x+3) = (x+3)^2 = 0$ (Factorizing: sum $= 6$ and product $= 9$).

$$x = -3$$

Hence, $x = -3$ is the only solution.

Example 2

Solve the quadratic equation $x^2 - 9 = 0$

Solution:

$(x-3)(x+3) = 0$ (Factorizing: sum $= 0$ and product$= 9$)

$$x = 3 \text{ or } x = -3$$

Hence, the solution of the quadratic equation is $x = 3,\ x = -3$.

Example 3

Solve the quadratic equation $4x^2 + 4x + 1 = 0$

Solution:

Re-writing $4x^2 + 2x + 2x + 1 = 0$

$$2x(2x+1) + 1(2x+1) = 0$$

$$(2x+1)(2x+1) = 0$$

$$(2x+1)^2 = 0$$

$$x = -\frac{1}{2}$$

Hence, the solution of the quadratic equation is $x = -\frac{1}{2}$.

Exercise 3.11

Solve the following quadratic equations using factorization method.

a. $x^2 + 10x + 25 = 0$

b. $x^2 - 8x + 16 = 0$

c. $x^2 - 4 = 0$

d. $9x^2 - 6x + 1 = 0$

Completing the square method

Example 1

Is it possible to solve $x^2 + 6x + 4 = 0$ using factorization method?

Solution:

Since there are no two integers whose sum is equal to 6 and product is equal to 4, this quadratic equation may not be solved using factorization method. Hence, we need another method to solve the equation.

$$x^2 + 6x + 9 + 4 - 9 = 0$$

$$(x+3)^2 - 5 = 0$$

$$(x+3)^2 = 5$$

$$x + 3 = \pm\sqrt{5}$$

$$x = -3 \pm \sqrt{5}$$

Completing the square is where we take the quadratic equation

$ax^2 + bx + c = 0, a \neq 0$ and convert it into $\left(x + \frac{b}{2a}\right)^2 + \frac{4ac-b^2}{4a^2} = 0$ as follows:

$x^2 + \frac{b}{a}x + \frac{c}{a} = 0$ ($since\ a \neq 0$)

$(x^2 + \frac{b}{a}x + \frac{b^2}{4a^2}) + \frac{c}{a} - \frac{b^2}{4a^2} = 0.$ Taking half of the coefficient of the middle term and squaring it and adding its opposite). The expression in the bracket is a perfect square. Hence, after simplifying, we have

$\left(x + \frac{b}{2a}\right)^2 + \frac{4ac-b^2}{4a^2} = 0 \ldots\ldots (*)$

Equivalently, $ax^2 + bx + c = 0$ if and only if $\left(x + \frac{b}{2a}\right)^2 + \frac{4ac-b^2}{4a^2} = 0$

Example 2

Solve $x^2 + 4x + 4 = 0$ using completing the square method.

Solution:

$(x^2 + 4x + 4) + 4 - 4 = 0$ (Adding the square of half of the coefficient of the middle term (4) and its opposite.

$$(x + 2)^2 - 0 = 0. \text{ (Writing as a perfect square)}$$

$$(x + 2)^2 = 0$$

$$x = -2$$

Example 3

Solve $x^2 + 6x + 7 = 0$ using completing the square method.

Solution:

$x^2 + 6x + 9 + 7 - 9 = 0$ (Adding the square of half of the coefficient of the middle term (6) and its opposite)

$(x^2 + 6x + 9) + 7 - 9 = 0$ (Colleting those terms which sum up as a perfect square)

$$(x + 3)^2 - 2 = 0. \text{ (Writing as a perfect square)}$$

$$(x + 3)^2 = 2$$

$$x + 3 = \pm\sqrt{2} \quad \text{(Taking the square root)}$$

$$x = -3 \pm \sqrt{2}$$

Therefore, $x = -3 + \sqrt{2}$ and $x = -3 - \sqrt{2}$ are the solutions.

Exercise 3.12

Use completing the square method to solve the following.

a. $x^2 + 4x + 1 = 0$

b. $x^2 - 6x - 5 = 0$

c. $2x^2 + 5x + 3 = 0$

The quadratic formula method

There are quadratic equations that cannot be solved by factorization method. This is generally true when the roots are not rational numbers. Consider the quadratic equation $ax^2 + bx + c = 0$, where $a \neq 0$.

From the completing the square method, we have $\left(x + \frac{b}{2a}\right)^2 + \frac{4ac - b^2}{4a^2} = 0$

Solving for x, $\left(x + \frac{b}{2a}\right)^2 = \frac{b^2 - 4ac}{4a^2}$

$$x + \frac{b}{2a} = \pm\sqrt{\frac{b^2 - 4ac}{4a^2}}$$

$$x = -\frac{b}{2a} \pm \frac{\sqrt{b^2 - 4ac}}{2a}$$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Quadratic formula

For the quadratic equation $ax^2 + bx + c = 0, a \neq 0$, $\boldsymbol{x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}}$

This is the quadratic formula

Example

Solve $x^2 - 6x + 3 = 0$ using the quadratic formula.

Solution:

$1, -6$ and 3 are the values for a, b, and c, respectively in the quadratic formula.

Thus,

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$x = \frac{-(-6) \pm \sqrt{(-6)^2 - 4(1)(3)}}{2(1)}$$

$$x = \frac{6 \pm \sqrt{36 - 12}}{2} = \frac{6 \pm \sqrt{24}}{2} = 3 \pm \sqrt{6}$$

Hence, $3 + \sqrt{6}$ and $3 - \sqrt{6}$ are the solutions. Can you solve this using other methods?

Exercise 3.13

Use quadratic formula to solve the following.

a. $x^2 + 3x + 1 = 0$ b. $x^2 + 5x - 2 = 0$ c. $2x^2 - 3x - 1 = 0$

Discriminant

Remark: When using the quadratic formula, you should be aware of three possibilities. These three possibilities are distinguished by a part of the formula called the discriminant. The discriminant is the value under the radical sign which is $b^2 - 4ac$. A quadratic equation with real numbers as coefficients can have the following:

The roots of the quadratic equation are $x = \frac{-b \pm \sqrt{D}}{2a}$, where $D = b^2 - 4ac$.

$If\ D > 0$, then the roots are real and distinct (unequal); the quadratic equation has two distinct roots.

$If\ D = 0$, then the roots are real and equal (coincident); the quadratic equation has exactly one real root.

$If\ D < 0$, then there are no real roots.

Example 1

Check whether $x^2 + 2x + 2 = 0$ has distinct real roots, one real root or no real roots. If root exists, find it.

Solution:

Here, $a = 1$, $b = 2$ and $c = 2$.

Since $b^2 - 4ac = 4 - 8 = -4 < 0$, no need to find the roots. The following is to show that the roots are not real number.

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$x = \frac{-(2)\pm\sqrt{(2)^2-4(1)(2)}}{2(1)}$$

$$x = \frac{-2\pm\sqrt{4-8}}{2} = \frac{-2\pm\sqrt{-4}}{2}$$ are not real numbers.

Example 2

Check whether $x^2 + 18x + 81 = 0$ has distinct real roots, one real root or no real roots. If root exists, find it.

Solution:

Here, $a = 1, b = 18$ *and* $c = 81$,

$D = b^2 - 4ac = 18^2 - 4(1)(81) = 324 - 324 = 0$. Since $D = 0$, the equation has exactly one root. Solving the equation,

$$(x + 9)^2 = 0$$

$$x = -9$$

Example 3

Check whether $x^2 + 4x + 3 = 0$ has distinct real roots, one real root or no real roots. If root exists, find it.

Solution:

Here, $a = 1, b = 4$ and $c = 3$, and then

$D = b^2 - 4ac = 4^2 - 4(3) = 16 - 12 = 4$

Since $D > 0$, the equation has two distinct real roots. Solving the equation,

$$(x + 1)(x + 3) = 0$$

$$x = -1 \text{ and } x = -3$$

Exercise 3.14

Check whether the following quadratic equations have two real roots, one real root or no real roots. If any real root exists, find it.

a. $x^2 + 2x + 3 = 0$ b. $x^2 + 12x + 36 = 0$ c. $x^2 + 8x + 7 = 0$

Relationships between roots and coefficients of a quadratic equation

If $r_1 = \frac{-b+\sqrt{b^2-4ac}}{2a}$ and $r_2 = \frac{-b-\sqrt{b^2-4ac}}{2a}$, then

$$r_1 + r_2 = \frac{-b+\sqrt{b^2-4ac}}{2a} + \frac{-b-\sqrt{b^2-4ac}}{2a} = \frac{-b}{a} \text{ and}$$

$$r_1 \cdot r_2 = \left(\frac{-b+\sqrt{b^2-4ac}}{2a}\right)\left(\frac{-b-\sqrt{b^2-4ac}}{2a}\right) = \frac{c}{a}$$

Example 1

Let $ax^2 + 6x + c = 0$. If the sum of the roots is $-\frac{6}{5}$ and the product is $\frac{1}{5}$, then find the values of a and c.

Solution:

Let r_1 and r_2 be the roots. From the relationships above,

$$r_1 + r_2 = -\frac{6}{a} = -\frac{6}{5}, \text{ which implies } a = 5$$

$$r_1 \cdot r_2 = \frac{c}{a} = \frac{1}{5} \qquad \text{Since } a = 5,\ \frac{c}{5} = \frac{1}{5}, \text{ which also implies } c = 1$$

Therefore, the quadratic equation is $5x^2 + 6x + 1 = 0$

Example 2

If the difference between the roots of the equation $x^2 - 13x + k = 0$ is 7, find the value of k.

Solution:

Let r_1 and r_2 be the roots. From the relationships above,

$$r_1 + r_2 = -\frac{b}{a} = -\frac{-13}{1} = 13 \text{ and}$$

$$r_1 - r_2 = 7$$

$$\begin{cases} r_1 + r_2 = 13 \\ r_1 - r_2 = 7 \end{cases}$$

Solving this, $r_1 = 10$ and $r_2 = 3$

$r_1 \cdot r_2 = (10)(3) = 30$, while $r_1 \cdot r_2 = \frac{c}{a} = \frac{k}{1}$ from the above relationship.

Therefore, $k = 30$.

Exercise 3.15

1. Let $x^2 + bx + c = 0$ If the sum of the roots of the quadratic equation is 10 and the product of the roots is 16, find b *and* c.
2. Find the quadratic equation: $ax^2 + 3x + c = 0$ when the sum of the roots is -3, the product is 2.
3. Solve the following using quadratic formula and check using any other methods of solving quadratic equation.

a. $x^2 - 16 = 0$

b. $x^2 - 9x = 0$

c. $3x^2 - 75 = 0$

d. $x^2 - 6x + 5 = 0$

e. $x^2 + 6x + 8 = 0$

f. $x^2 + 3x - 28 = 0$

g. $2x^2 + 7x + 3 = 0$

h. $-x^2 + 2x + 15 = 0$

i. $x^2 + 10x + 7 = 0$

j. $-3x^2 + 10x - 7 = 0$

k. $3x^2 = 6x - 1$

l. $5x^2 + x + 2 = 0$

m. $-3x^2 + 8x - 5 = 0$

n. $2x^2 + 5x = 12$

o. $8x^2 - 10x = -3$

p. $2x^2 + 5x + 2 = 0$

q. $-3x^2 + 7x - 6 = 0$

r. $10x^2 - 9x + 6 = 0$

s. $3x^2 = -x + 14$

t. $9x^2 - 7 = 9x$

u. $\frac{1}{3}x^2 - x - 2 = 0$

w. $-4x(x - 2) = 6(x + 3) - 11x^2$

x. $x(x - 3) = -7 - 10x$

3.3.3 Equations involving exponents and radicals

Equations involving exponents 1

Activity 3.6

Consider the following equations and discuss on their solutions:

a. $x^2 = 4$ b. $x^3 = -64$

How many solution(s) does each equation have? What are they? What can you generalize?

Note

For a > 0, $a^x = a^y$ if and only if $x = y$.

Example 1

Solve the followings.

a. $2^x = 8$

b. $3^x = \frac{1}{27}$

Solution:

a. $2^x = 2^3$ (Making the same base by writing $8 = 2^3$)

$x = 3$ (Equating the exponents because the bases are the same)

b. $3^x = \frac{1}{3^3}$

$3^x = 3^{-3}$

$x = -3$

Some rules of exponential equations.

(a, m, and n are real number and $a \neq 0$, and $b \neq 0$)

a. $\frac{1}{a^n} = a^{-n}$

b. $a^m a^n = a^{m+n}$

c. $(a^m)^n = a^{mn}$

d. $(ab)^n = a^n b^n$

e. $\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}$

f. $\frac{a^m}{a^n} = a^{m-n} = \frac{1}{a^{n-m}}$

Exercise 3.16.1

Solve the following equations.

a. $3^x = 81$ b. $5^x = 125$ c. $2^x = \frac{1}{8}$ d. $4^x = \frac{1}{64}$

Example 2

Solve the following equations.

a. $8^x = 16$ b. $2^{x+4} = 32$ c. $9^{x+4} = 27$

Solution:

a. $2^{3x} = 2^4$

$3x = 4$ (Equating the exponents)

$x = \frac{4}{3}$

b. $2^{x+4} = 32$

$2^{x+4} = 2^5$

$x + 4 = 5$

$x = 1$

c. $9^{x+4} = 27$

$3^{2(x+4)} = 3^3$

$2x + 8 = 3$

$x = -\frac{5}{2}$

Exercise 3.16.2

Solve the following.

a. $4^x = 32$

b. $3^{2x-1} = 243$

c. $32^x = 2^{x+4}$

Equations involving exponents 2

Example 1

Solve: $(x^2 + 6x)^{\frac{1}{4}} = 2$

Solution:

Remove the 4th root by raising each side of the equation to the 4th power.

$$((x^2 + 6x)^{\frac{1}{4}})^4 = 2^4$$

$$x^2 + 6x = 16$$

$$x^2 + 6x - 16 = 0$$

$$(x + 8)(x - 2) = 0$$

Therefore, $x = -8$, $x = 2$.

These are possible solution. Verify them as follows.

If $x = -8$,

$(LHS) = [(-8)^2) + 6(-8)]^{\frac{1}{4}} = (64 - 48)^{\frac{1}{4}} = (16)^{\frac{1}{4}} = (2^4)^{\frac{1}{4}} = 2$

Thus, $(LHS) = (RHS)$. Therefore $x = -8$ is a solution.

Similarly, if $x = 2$, then

$(LHS) = [(2)^2) + 6(2)]^{\frac{1}{4}} = (4 + 12)^{\frac{1}{4}} = (16)^{\frac{1}{4}} = (2^4)^{\frac{1}{4}} = 2 = (RHS)$

Therefore, $x = 2$ is also a solution.

Example 2

Solve for y if $5y^{\frac{2}{3}} + 3 = 23$.

Solution:

Isolate the y-term on the left side of the equation. Subtract 3 from each side of the equation.

$$5y^{\frac{2}{3}} = 23 - 3$$

$$5y^{\frac{2}{3}} = 20$$

$$y^{\frac{2}{3}} = 4.$$

First, raise both sides by 3rd power,

$$\left(y^{\frac{2}{3}}\right)^{3} = 4^{3}$$

Then $y^{2} = 4^{3} = (2^{2})^{3} = (2^{3})^{2}$,

$$y = \pm\sqrt{(2^{3})^{2}} = \pm 2^{3} = \pm 8$$

Exercise 3.17

1. Solve the following equations involving exponents.

   a. $(x^{2} + 2x)^{\frac{1}{2}} = 4$ b. $\left(16x^{\frac{1}{2}}\right)^{\frac{1}{4}} = 2$

2. Solve the following equations involving exponents.

   a. $2^{x} = 2$ b. $3^{x} = 9$ c. $25^{2x-1} = 125^{3x+4}$

   d. $\left(\frac{1}{2}\right)^{x} = 16$ e. $9^{2x-5} = 27$ f. $\left(\frac{3}{2}\right)^{x} = \frac{81}{16}$

   g. $2^{x+6} = 32$ h. $x^{\frac{2}{5}} = 4$ i. $8^{2x-3} = \left(\left(\frac{1}{16}\right)^{x-2}\right)$

Equations involving radicals

Definition 3.4

Radical equations are equations that contain variables in the radicand (the expression under a radical symbol).

For example, $\xi\,\overline{3x+18} = x$, $\xi\,\overline{x+3} = x-3$ and $\xi\,\overline{x+5} - \xi\,\overline{x-3} = 2$ are radical equations.

In solving radical equations, the following are steps to follow.

1. Positive radical number and radicand (the expression under a radical symbol) should be positive. Thus, at first, specify the domain of x. That is, $\xi\,\overline{a} > 0,\ a > 0$

2. Isolate the radical expression on one side of the equal sign. Put all remaining terms on the other side.
3. If the radical is a square root, then square both sides of the equation.
4. Solve the resulting equation.
5. If a radical term still remains, repeat steps 1–2.
6. Check solutions by substituting them into the original equation.

Example 1

Solve the equation $\sqrt{15-2x}=x$.

Solution:

First specify the domain of x.

Since the radicand term $(15-2x)$ should be non-negative, that is $15-2x\geq 0$.

Thus, the right side should also be non-negative too. $x\geq 0$

From the 1st inequality, $2x\leq 15$ then $x\leq \frac{15}{2}$

Hence, $0\leq x\leq \frac{15}{2}$

Squaring both sides of the given equation,

$$15-2x=x^2$$

$$x^2+2x-15=0$$

$$(x-3)(x+5)=0$$

$$x-3=0 \text{ or } x+5=0$$

$$x=3 \text{ or } x=-5$$

Observe that $x=-5$ cannot be a solution. (Why? Because $0\leq x\leq \frac{15}{2}$)

Example 2

Solve the equation: $\sqrt{x+3}=3x-1$

Solution:

First specify the domain of x.

$$x + 3 \geq 0 \quad \text{and} \quad 3x - 1 \geq 0$$

$$x \geq -3 \ \text{ and } x \geq \frac{1}{3}, \quad \text{Thus, } x \geq \frac{1}{3}$$

Squaring both sides of the given equation,

$$x + 3 = 9x^2 - 6x + 1$$

$$9x^2 - 7x - 2 = 0$$

$$(x - 1)(9x + 2) = 0 \quad \text{(after factorizing)}$$

$$x = 1 \ \text{ or } \ x = -\frac{2}{9}$$

Observe that $x = 1$ is the only solution, as $x \geq \frac{1}{3}$.

Example 3

Solve the equation: $\sqrt{2x + 3} + \sqrt{x - 2} = 4$

Solution:

First specify the domain of x.

$2x + 3 \geq 0$ and $x - 2 \geq 0$ From the two inequalities, $x \geq 2$……(i)

In addition, $4 - \sqrt{x - 2} \geq 0$ Then, $\sqrt{x - 2} \leq 4$……(ii)

The equation becomes $\sqrt{2x + 3} = 4 - \sqrt{x - 2}$

Squaring both sides of the equation,

$$2x + 3 = 16 - 8\sqrt{x - 2} + x - 2$$

$$x - 11 = -8\sqrt{x - 2}$$

Squaring both sides of the equation again,

$$x^2 - 22x + 121 = 64(x - 2)$$

$$x^2 - 86x + 249 = 0$$

$$(x - 3)(x - 83) = 0 \quad \text{(After factorizing)}$$

$$x = 3 \ \text{ or } \ x = 83$$

Observe that $x = 3$ is the only solution, while $x =$ 83 cannot be a solution, because $x = 3$ meet both conditions (i) and (ii), however, $x = 83$ does not.

Exercise 3.18

Solve the following equations involving radicals.

a. $\sqrt{3x-2}=x$ b. $\sqrt{3x-2}=\sqrt{4x+2}$

c. $\sqrt{3x+7}+\sqrt{x+2}=1$

3.4 Some Applications of Solving Equations

Application of the system of linear equations in two variables

Activity 3.7

Meal tickets at a wedding ceremony cost Birr 4.00 for children and Birr 12.00 for adults. If 1,650 meal tickets were bought for a total of Birr 14,200, how many children and how many adults bought meal tickets?

Example

The cost of a ticket to a circus is Birr 25.00 for children and Birr 50.00 for adults. On a certain day, attendance at the circus is 2,000 and the total gate revenue is Birr 70,000. How many children and how many adults bought tickets?

Solution:

Let $c=$ the number of children and $a=$ the number of adults in attendance.

The total number of people is 2,000. We can use this to write an equation for the number of people at the circus that day.

$$c+a=2{,}000$$

The money collected from all children can be found by multiplying Birr 25.00 by the number of children, $25c$. The money collected from all adults can be found by multiplying Birr 50.00 by the number of adults, $50a$. The total revenue is Birr 70,000. We can use this to write an equation for the revenue.

$$25c + 50a = 70{,}000$$

We now have a system of linear equations in two variables.

$$c + a = 2{,}000$$

$$25c + 50a = 70{,}000$$

In the first equation, the coefficient of both variables is 1. We can quickly solve the first equation for either c *and* a. Let us solve for a.

$$a = 2{,}000 - c$$

Substitute the expression $2{,}000 - c$ in the second equation for a and solve for c.

$$25c + 50(2{,}000 - c) = 70{,}000$$

$$-25c = -30000$$

$$c = 1200$$

Substitute $c = 1200$ into the first equation to solve for a.

$$1200 + a = 2000$$

$$a = 800$$

We find that 1,200 children and 800 adults bought tickets to the circus that day.

Exercise 3.19

1. The cost of a ticket to a football match is Birr 50.00 for children and Birr 75.00 for adults. On a certain day, attendance at the football match is 25,000 and the total gate revenue is Birr 1,375,000. How many children and adults bought tickets?
2. The cost of two tables and three chairs is Birr 705. If the table costs Birr 40 more than the chair, find the cost of the table and the chair.
3. Abdullah is choosing between two car-rental companies. The first, "Keep on carefully driving", charges an up-front fee of Birr 15 and 61 cents a kilometer. The second, "Atiften Tidesaleh", charges an up-front fee of Birr 12 and 65 cents a kilometer. When will "Keep on carefully driving" be better choice for Abdullah?

Application of quadratic equations

Example 1

An object dropped from a height of 600m has a height, h, in meter after t seconds have elapsed, such that $h = 600 - 16t^2$. Find the time taken to reach a height of $200m$ by first finding an expression for $t > 0$.

Solution:

We are asked two things. First, we will solve the height equation for t, and then we will find how long it takes for the object's height above the ground to be $200m$. Solve for t:

$$h = 600 - 16t^2$$

$$t^2 = \frac{600-h}{16}$$

$$t = \sqrt{\frac{600-h}{16}} = \frac{\sqrt{600-h}}{4}$$

We want time to be only positive since we are talking about a measurable quantity, so we will restrict our answers to just $t = \frac{\sqrt{600-h}}{4}$

We want to know at what time the height will be $200m$. So we can substitute 200 for h.

$$t = \frac{\sqrt{600-200}}{4} = \frac{\sqrt{400}}{4} = \frac{20}{4} = 5$$

Therefore, it takes 5 seconds for the object to be at the height of $200m$.

Example 2

Suppose a rocket projectile is launched from a tower into the air with an initial velocity of 48 meters per second. Its height, h, in meter above the ground is modeled by the equation $h = -16t^2 + v_0t + 64$ where $t > 0$ is the time in second, since the projectile was launched and v_0 is the initial velocity. How long was the projectile in the air?

Solution:

The rocket projectile was in the air till the height $h = 0$, with initial velocity $v_0 = 48m/s$

$$0 = -16t^2 + 48t + 64$$
$$0 = 16(t^2 - 3t - 4)$$
$$0 = 16(t + 1)(t - 4)$$

Hence, $t = 4$ seconds. , $t = -1$ cannot be taken as a solution. Why? This means that the rocket projectile hits the ground after 4 seconds.

Exercise 3.20

A ball is shot into the air from the ground. Its initial velocity is 32 meter per second. The equation $h = -16t^2 + 32t + 48$ can be used to model the height of the ball after t seconds. About how long does it take for the ball to hit the ground?

Summary

1. Equations are expressions involving equality.
2. Linear equation in one variable is of the form $ax + b = 0$, where x is the variable and a and b are real coefficients and $a \neq 0$.
3. An equation of the type $cx + dy = 0$, where c and d are arbitrary constants $c \neq 0$ and $d \neq 0$ is said to be a linear equation in two variables and its solutions are infinitely many points and the graph is a straight line.
4. A system of linear equation is a set of two or more linear equations, and a system of two linear equations in two variables are equations that can be represented as

$$\begin{cases} a_1x + b_1y = c_1 \\ a_2x + b_2 = c_2 \end{cases}$$

5. Solution to a system of linear equation in two variables is the set of ordered pairs (x, y) that satisfy both linear equations.
   a. If $\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$, then the system has infinite solutions and is called dependent system.
   b. If $\frac{a_1}{a_2} = \frac{b_1}{b_2} \neq \frac{c_1}{c_2}$, then the system has no solution and is called inconsistent system.
   c. If $\frac{a_1}{a_2} \neq \frac{b_1}{b_2}$, then the system has one solution and is called independent system.
6. Geometrically,
   a. The system has infinite solution if the two lines coincide (or identical).
   b. The system is said to have no solution if the lines are parallel, i.e. no intersection point.
   c. The system is said to have one solution if the lines intersect at one point.

7. A system of linear equation in two variables can be solved by one of the following methods: graphically, or by substitution or by elimination.
8. The absolute value of a number x, denoted by $|x|$, is defined as the distance $|x|$ from zero on a number line.
9. For any non- negative number p, $|x| = p$ means $x = p$ or $x = -p$.
10. An equation of the form $ax^2 + bx + c = 0$ where $a, b, c, \in R$ and $a \neq 0$ is a quadratic equation.
11. Factorization, completing the square and quadratic formula are methods which can be used to solve a quadratic equation $ax^2 + bx + c = 0$
12. If the quadratic equation, $ax^2 + bx + c = 0$, has roots $\frac{-b\pm\sqrt{b^2-4ac}}{2a}$, then the sum and product of the roots are $-\frac{b}{a}$ and $\frac{c}{a}$, respectively.
13. The discriminant of quadratic equations, $D = b^2 - 4ac$,

    $If\ D > 0$, then the quadratic equation has two distinct roots.

    $If\ D = 0$, then the quadratic equation has exactly one real root.

    $If\ D < 0$, then there are no real roots.
14. Let $x^n = k$. If n is even, $x = \pm\sqrt[n]{k}$. If n is odd, $x = \sqrt[n]{k}$.
15. For $a > 0, a^x = a^y$ if and only if $x = y$.
16. For solving radical equations, specify the domain of x, as the radicand (the expression under a radical symbol) should be positive.

Review Exercise

For questions 1-3, choose the correct answer from the given alternatives.

1. Which of the following linear equation passes through the origin?

   a) $y = 3x + 0$ b) $y = 3x + 1$ c) $y = 3x - 10$ d) $y = 3x - \frac{1}{2}$

2. Which of the following is the solution of the linear equation $x + 2 = -x - 4$

   a) $x = 3$ b) $x = -3$ c) $x = -10$ d) $x = -\frac{1}{2}$

3. Which of the following is the solution of $-2x + 2 = x + 4 - 3x$

a) $x = 1$ b) $x = 0$ c) $x = -1$ d) No solution

For questions 4-9 use any Method to find the solution to the given system or to determine if the system is inconsistent or dependent

4. $\begin{cases} x - 7y = -11 \\ 5x + 2y = -18 \end{cases}$

5. $\begin{cases} -4x + 2y = 3 \\ 2x - y = -12 \end{cases}$

6. $\begin{cases} 3x + 9y = -6 \\ -4x - 12y = 8 \end{cases}$

7. $\begin{cases} 6x - 5y = 8 \\ -12x + 2y = 0 \end{cases}$

8. $\begin{cases} -x + 5y = 2 \\ 5x - 25y = -10 \end{cases}$

9. $\begin{cases} 2x + 3y = 20 \\ x + \frac{3}{2}y = 10 \end{cases}$

For questions 10-39, solve each of the following non-linear equations.

10. $|4x - 7| = 5$
11. $|3 - 4x| = 5$
12. $|2 - 4x| = 2$
13. $|\frac{2}{3}x - 7| = 0$
14. $|x^2 - 1| = 3$
15. $|x^2 + 9| = 1$
16. $x^2 - 3x = 0$
17. $x^2 + 9x = 0$
18. $x^2 + 8x + 15 = 0$
19. $x^2 - 5x + 7 = 0$
20. $-3x^2 + x + 4 = 0$
21. $x^2 + 6x + 9 = 0$
22. $2x^2 + 5x + 3 = 0$
23. $2x^2 - 5x + 2 = 0$
24. $6x^2 - x - 1 = 0$
25. $-5x^2 + 6x - 1 = 0$
26. $2x^2 - 11x + 14 = 0$
27. $2x^2 - 7x - 1 = 0$
28. $16x - 7x^2 = 79$
29. $4^{2x+5} = 64$
30. $9^{x+1} = 81$
31. $64 \cdot 4^{3x} = 16$
32. $2^{-2x+5} \cdot \frac{1}{4} = 2^{x+2}$
33. $6^{2x-3} = 6$
34. $(x - 4)^{2/3} = 25$
35. $(x + 5)^{3/2} = 8$
36. $(x + 12)^{3/2} = 8$
37. $\sqrt{3x + 18} = x$
38. $\sqrt{x + 3} = x - 3$
39. $\sqrt{x + 5} - \sqrt{x - 3} = 2$

For questions 40-54, solve the following word problems.

40. If $x = 2$ and $y = 3$ is a solution of the equation $8x - ay + 2a = 0$, find the value of a.

41. If the sum of two numbers is 13 and their product is 42 find the numbers.

42. The present age of Gaddisa is 4 years more than twice the present age of his cousin Chala. Chala's present age is 2 years more than $\frac{1}{3}$ of the present age of Gaddisa. Find the difference of their ages.

43. We want to fence a field whose length is three times the width and we have 100 meters of fencing material. If we use all the fencing material, what would the dimensions of the field be?

44. Three coffees and two makiatos cost a total of Birr 17. Two coffees and four makiatos cost a total of Birr 18. What is the individual price for a single coffee and a single makiato?

45. Which of the following is the solution for the system of linear equations

$$\begin{cases} 3x + y = 11 \\ y = x + 2 \end{cases} ?$$

A) $x = 3,\ y = 2$ B) $x = 3,\ y = 5$

C) $x = \frac{9}{4},\ y = \frac{17}{4}$ D) $x = \frac{17}{4},\ y = \frac{9}{4}$

46. The solution for the quadratic equation $x^2 + 10x = -25$ is:

A) 10 B) -5 C) -10 D) 5

47. The solution for the absolute value equation $|-2x + 8| = 12$ is:

A) $x = 2,\ x = 10$ B) $x = -2,\ x = 10$

C) $x = 2,\ x = -10$ D) $x = -2,\ x = -10$

48. Find the values of k for which the quadratic expression $(x - k)(x - 10) + 1 = 0$ has integral roots.

49. Find the values of k such that the equation $\frac{p}{x+r} + \frac{q}{x-r} = \frac{k}{2x}$ has two equal roots.

50. Find the quadratic equation with rational coefficients when one root is $\frac{1}{2+\sqrt{5}}$.

51. If the coefficient of x in the quadratic equation $x^2 + bx + c = 0$ was taken as 17 in place of 13, its roots were found to be -2 and -15. Find the roots of the original quadratic equation.

52. For what value of k, both the quadratic equations $6x^2 - 17x + 12 = 0$ and $3x^2 - 2x + k = 0$ will have a common root.

53. Solve $|x^2 + 2x - 4| = 4$

UNIT 4

SOLVING INQUALITIES

Unit Outcomes

By the end of this unit, you will be able to:

- **Solve linear inequalities in one variable.**
- **Solve inequalities involving absolute value.**
- **Solve system of linear inequalities.**
- **Solve quadratic inequalities.**
- **Apply inequalities in real life situations.**

Unit Contents

KEY Words

- linear equation
- Quadratic inequality
- linear inequality
- sign chart
- Absolute value
- open interval
- solution set
- closed intervals
- Product property
- Quadratic equation
- discriminant
- complete listing

INTRODUCTION

An algebraic inequality is a mathematical sentence connecting an expression to a value, variable, or another expression with an inequality sign (<, ≤, >, or ≥). Solving inequalities is similar to solving equations.

4.1 Revision on Linear Inequalities in one Variable

Activity 4.1

a. If you add or subtract the same number from both sides of an inequality, will the inequality sign remain the same or have to be reversed? For instance, if $2 < 5$, then can you conclude that $2 + c < 5 + c$? Why? $2 - c < 5 - c$? Why?

b. If you multiply or divide by a negative number, will the inequality sign change? Why?

Review on some properties of inequalities.

a. If $\mathrm{a} > \mathrm{b}$, then $\mathrm{a} + \mathrm{c} > \mathrm{b} + \mathrm{c}$

b. If $\mathrm{a} > \mathrm{b}$, then $\mathrm{a} - \mathrm{c} > \mathrm{b} - \mathrm{c}$

c. If $\mathrm{a} > \mathrm{b}$ and $\mathrm{m} > 0$, then $\mathrm{ma} > \mathrm{mb}$ and $\frac{\mathrm{a}}{\mathrm{m}} > \frac{\mathrm{b}}{\mathrm{m}}$

d. If $\mathrm{a} > \mathrm{b}$ and $\mathrm{m} < 0$, then $\mathrm{ma} < \mathrm{mb}$ and $\frac{\mathrm{a}}{\mathrm{m}} < \frac{\mathrm{b}}{\mathrm{m}}$

Example 1

Solve the following inequalities.

a. $x - 5 > 2$ **b.** $2x > 4$ **c.** $\frac{1}{3}x < 1$

d. $-3x > 6$ **e.** $2x - 3 \geq 1$

Solution:

a. $x - 5 > 2$

$x > 2 + 5$

$x > 7$

b. $2x > 4$

$x > \frac{4}{2}$

$x > 2$

c. $\frac{1}{3}x < 1$

$x < 3$

d. $-3x > 6$

$x < -2$

e. $2x - 3 \geq 1$

$2x \geq 4, \ x \geq 2$

Exercise 4.1.1

Solve the following inequalities.

a. $x - 1 > 3$ b. $4x < 12$

c. $-\frac{1}{5}x \leq 2$ d. $3x + 1 \geq 7$

Example 2

Express the following using the notation of intervals.

a

b

c

Solution:

a. $[1, 6]$ **b.** $[-3, 2)$ **c.** $(a - 2, a + 1]$

Note

The symbol $[a, b]$ is closed interval with end-points a and b. It shows the set of all real numbers x such that $a \leq x \leq b$.

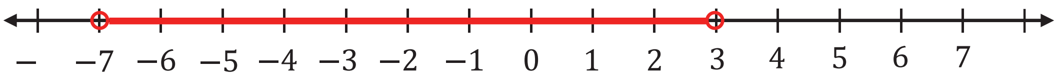
The symbol (a, b) is open interval with end-points a and b. It shows the set of all real numbers x such that $a < x < b$.

The symbol $[a, b)$ and $(a, b]$ is half-open interval. They show the set of all real numbers x such that $a \leq x < b$ and $a < x \leq b$, respectively.

Example 3

Write down $(-7, 3)$ on the number line

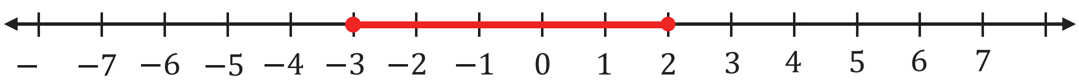
Solution:



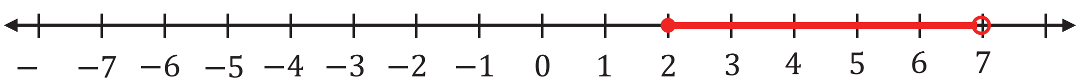
Exercise 4.1.2

1. Express the following using the notation of intervals.

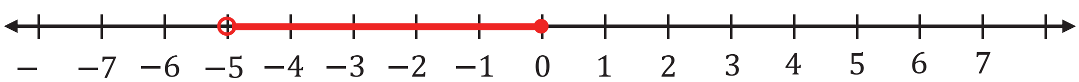
   a.



   b.



   c.



2. Express the following intervals on a number line.

   a. $(-3,1)$ b. $[-2,4)$ c. $[-5,1]$

Example 1

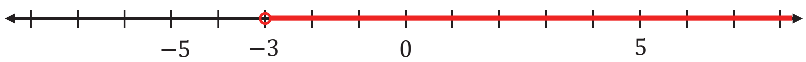
Solve the inequality below and express it on a number line.

$$x + 1 > -2$$

Solution:

$$x + 1 > -2$$

$$x > -3$$



Using interval notation, the solution is $(-3, \infty)$. To represent the solution on a number line, we place the solution (boundary point) on the number line. Since $x = -3$ is not a solution to the inequality $(<)$ we use an open dot as shown. Using the set notation, the solution is $\{x \mid x > -3\}$

Example 2

Solve $3x - (x + 2) \geq 0$, and express it on a number line

Solution:

$$3x - x - 2 \geq 0$$

$$2x \geq 2$$

$$x \geq 1 \text{ (dividing both sides by 2)}$$

Now, we place the solution (boundary point) on the number line. Since, $x = 1$ is also a solution to the inequality $(\geq)$ we use a filled-in dot as follows:

The solution is $x \geq 1$. Using interval notation, the solution is $[1, \infty)$.

Example 3

Solve $x + 2 < -x + 6$, and express it on a number line

Solution:

$$x + x < 6 - 2$$

$$2x < 4$$

$$x < 2$$

Using interval notation, the solution is $(-\infty, 2)$. The solution is represented on a number line as follows:

Example 4

Solve the inequality below and graph the solution set and write it in an interval notation.

$$-\frac{2}{3}x + \frac{1}{2} \leq \frac{5}{6}$$

Solution:

$$-\frac{2}{3}x + \frac{1}{2} \leq \frac{5}{6}$$

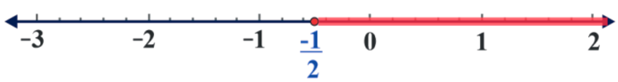
$$6\left(-\frac{2}{3}x + \frac{1}{2}\right) \leq 6\left(\frac{5}{6}\right)$$

$$-4x + 3 \leq 5$$

$$-4x \leq 2$$

$$x \geq -\frac{1}{2} \text{ (dividing both sides by } -4\text{)}$$

The solution in interval notation is $\left[-\frac{1}{2}, \infty\right)$



Exercise 4.2

1. Solve the inequality below, express it on a number line, and write it in an interval notation.

   a. $x - 2 > 3$

   b. $x + 1 \leq 5$

2. Solve the following linear inequalities.

a. $x + 3 < 5$

b. $x + 6 > 2$

c. $x - 10 \leq -7$

d. $3x - (2x + 2) \leq 7$

e. $\frac{1}{3}x \geq 4$

f. $\frac{2}{3}x \leq 6$

g. $-5x > -15$

h. $4x - (2x + 8) \geq 0$

i. $5x - 3(x - 1) < 5$

j. $8x - 5 > 2x + 4$

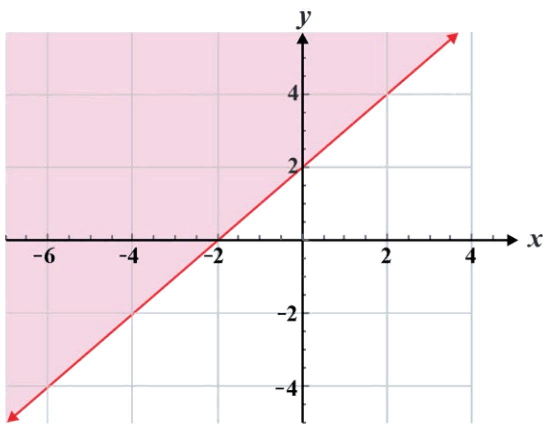
k. $\frac{x+2}{5} \geq \frac{2x}{3}$

l. $\frac{1}{4}x + 6 > 5$

4.2 Systems of Linear Inequalities in Two variables

Activity 4.2

1. How do you draw the graph of a linear equation given the y-intercept and the slope?
2. What mathematical statement describe all the points above the line $y = x + 3$, below the line $y = x + 3$, on the line $y = x + 3$?
3. What linear inequality is represented by the graph below?



a. $x - y \geq 2$

b. $x - y \leq 2$

c. $-x + y \geq 2$

d. $-x + y \leq 2$

Definition 4.1

Linear inequalities in two variables represent the unequal relation between two algebraic expressions that includes two distinct variables.

Important procedures in solving inequalities in two variables graphically

To solve an inequality in two variables graphically, first sketch the corresponding equations on the same xy-plane. This graph is the boundary line (or curve); since all points that make the inequality true lie on one side or the other of the line. Before graphing the equations, decide whether the line or curve is part of the solution or not, that is, whether it is solid or dashed.

If the inequality symbol is either $\leq$ or $\geq$, then the points on the boundary line are solutions to the inequality and the line must be solid. If the symbol is either $<$ or $>$, then the points on the boundary line are not solutions to the inequality and the line is dashed.

Next, decide which side of the boundary line must be shaded to show the part of the graph that represents all (x, y) coordinate pairs that make the inequality true. To do this, choose a point not on the boundary line. Substitute the x- and y-values of this point into the original inequalities. If the inequality is true for the test point, then shade the region on the side of the boundary line that contains the test point. If the inequality is false for the test point, then shade the opposite region. The shaded portion represents all of the (x, y) coordinate pairs that are solutions to the original inequality.

Example

Solve the inequality $y > -x$ using graphical method.

Solution:

Sketch the corresponding equation $y = -x$ on the xy-plane.

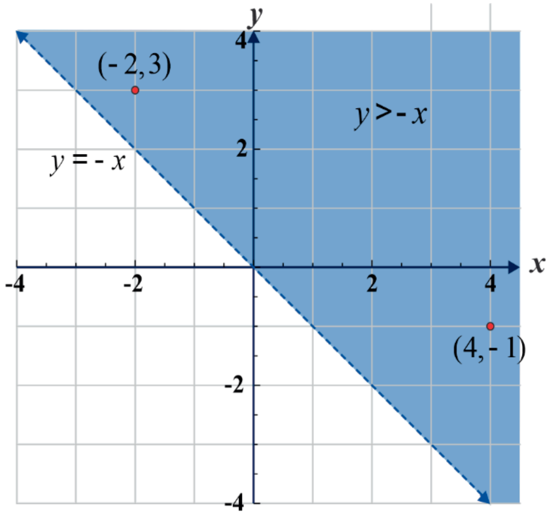


Figure 4.1

You can check a couple of points to determine which side of the boundary line to shade. Checking points $(-2,3)$ and $(4,-1)$ yield true statements. So, we shade the area above the line. The line is dashed as points on it are not part of the solution.

Exercise 4.3

Solve the following linear inequalities in two variables using graphical method.

a. $y \leq -x$

b. $y > x$

c. $y < x + 2$

d. $y \geq -x + 1$

e. $x > -y$

Solving systems of linear inequalities in two variables using graphical method

Activity 4.3

By drawing the paired inequalities on the same xy-plane, identify the common region

a. $y > x$ and $y > -x$

b. $y < x + 2$ and $y > 2x$

The solution to the system of inequalities is the overlap of the shading from the individual inequalities. When two boundary lines are graphed, there are often four regions. The region containing the coordinate pairs that make both of the inequalities true is the solution region.

Example 1

Solve the system of linear inequalities $\begin{cases} y < 2x + 5 \\ y > -x \end{cases}$ graphically

Solution:

First, draw the graph of the linear equation, $y = 2x + 5$. As shown in Fig 4.2, the dashed line is $y = 2x + 5$. Every ordered pair in the shaded area below the line is a solution to $y < 2x + 5$, as all the points below the line will make the inequality true. To confirm this, you may substitute x and y coordinate of points such as $(3, 1)$, $(-1, -1)$, etc. into the inequalities. So, the shaded area (Fig 4.2) shows all the solutions for this inequality. The boundary line divides the coordinate plane in half. In this case, it is shown as a dashed line as the points on the line do not satisfy the inequality.

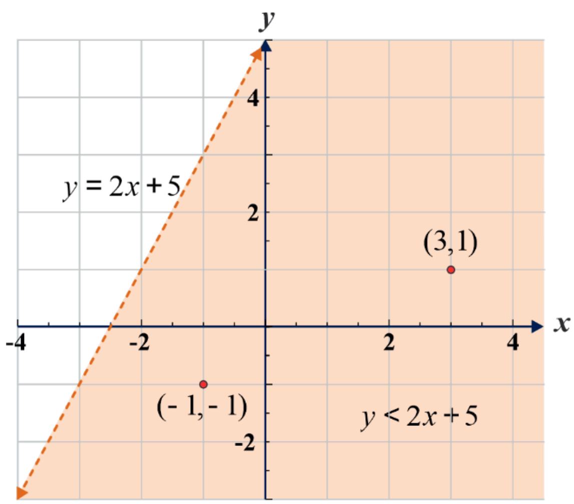


Figure 4.2

If the inequality had been $y \leq 2x + 5$, then the boundary line would have been solid. Next, draw the graph of $y > -x$ in the same plane. As shown in Fig 4.3, the area shared by light blue shows the region $y > -x$. The points on the line $y = -x$ do not satisfy $y > -x$, the line was dashed.

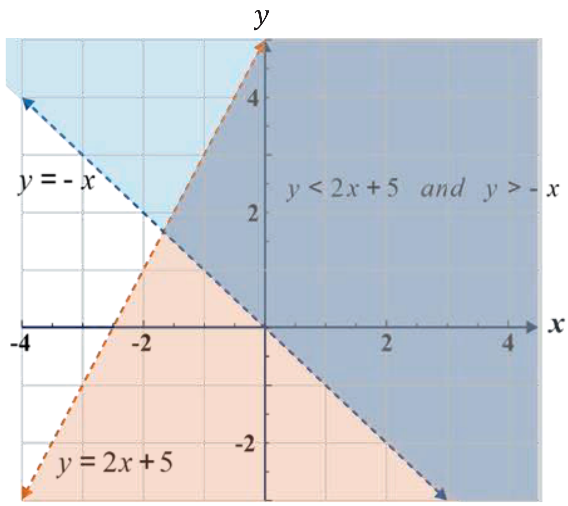


Figure 4.3

The dark blue area (Fig 4.3) shows where, the solutions of the two inequalities overlap. This area is the solution to the ***system of inequalities***. Any point within this blue region will be true for both $y < 2x + 5$ and $y > -x$.

Exercise 4.4

Solve the following linear inequalities in two variables using graphical method.

a. $\begin{cases} y < 2x + 1 \\ y > -x \end{cases}$ b. $\begin{cases} y > 2x + 1 \\ y < -x \end{cases}$ c. $\begin{cases} y < x + 1 \\ y > -3x \end{cases}$

d. $\begin{cases} y > x + 1 \\ y > -3x \end{cases}$ e. $\begin{cases} y < -x + 2 \\ y < x \end{cases}$

Example 2

Solve the system of linear inequalities $y \leq x - 2$ and $y > -3x + 5$ using graphical method.

Solution:

First, graph the inequality $y \leq x - 2$. The related equation is $y = x - 2$ (Fig. 4.4). Since the inequality is $\leq$ (less than or equal to), any points on the line satisfy the inequality. Thus, you use the solid line for the border line.

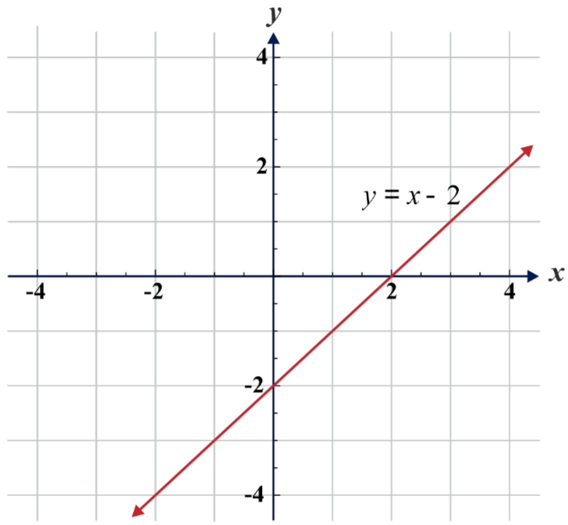


Figure 4.4

Consider a point that is not on the line, for instance, $(0, 0)$, and substitute in the inequality.

$y \leq x - 2$

$(\text{L.H.S}) = 0, \ (\text{R.H.S}) = 0 - 2 = -2$

Thus, $(\text{L.H.S}) \geq (\text{R.H.S})$. This is not true. So, the solution does not contain the point $(0, 0)$. Hence, shade the lower half of the line $y = x - 2$ (Fig 4.5).

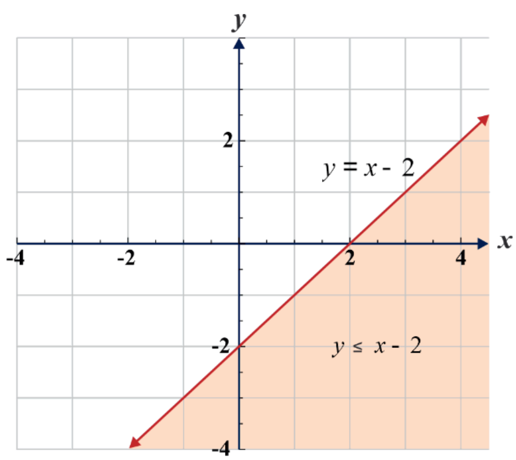


Figure 4.5

Similarly, draw a dashed line for the related equation $y > -3x + 5$ of the second inequality which has a strict inequality (Fig 4.6). The point (0,0) does not satisfy the inequality, so shade the half that does not contain the point (0,0).

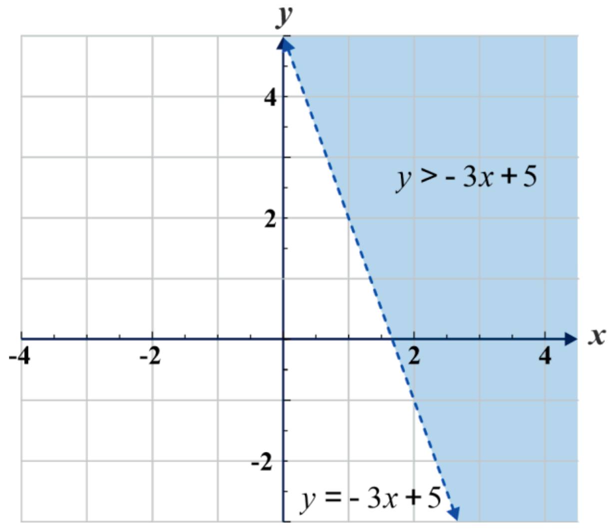


Figure 4.6

The solution of the system of inequalities is the intersection region of the solutions of the two inequalities (Fig 4.7).

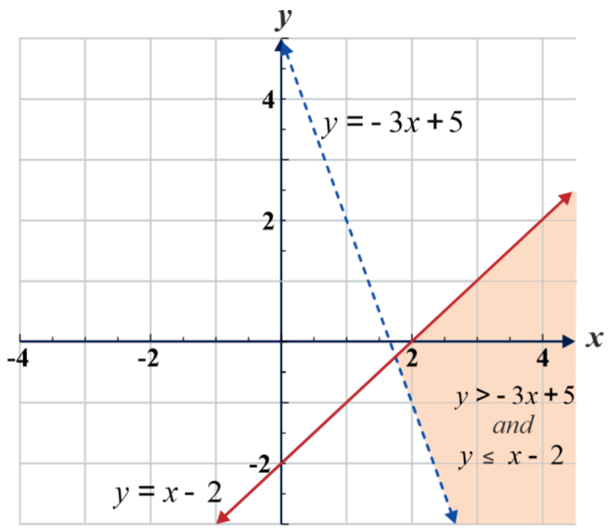


Figure 4.7

Example 3

Solve the following $\begin{cases} 2x + 3y \geq 12 \\ 8x - 4y > 1 \\ x < 4 \end{cases}$

Solution:

Rewrite the first two inequalities with y alone on one side.

$$3y \geq -2x + 12$$

This leads to $y \geq -\frac{2}{3}x + 4$.

Similarly, the second inequality,

$8x - 4y > 1$, $4y < 8x - 1$, then,

$y < 2x - \frac{1}{4}$. Now, graph the inequality $y \geq -\frac{2}{3}x + 4$. The related equation is $y = -\frac{2}{3}x + 4$. Since the inequality is $\geq$, the border line is solid. Shade the upper half of the line (Fig 4.8), as

$y \geq -\frac{2}{3}x + 4$.

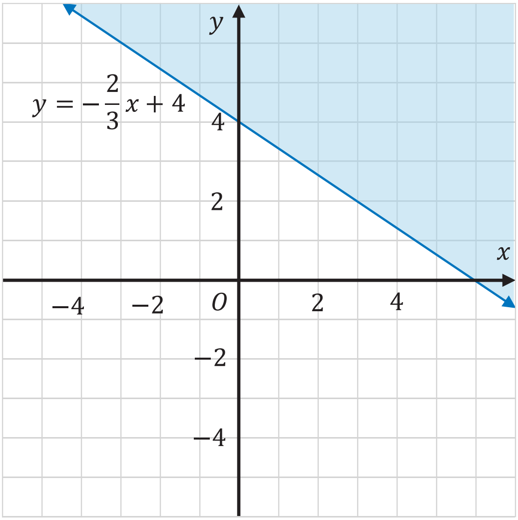


Figure 4.8

Similarly, draw a dashed line of related equation $y = 2x - \frac{1}{4}$ of the second inequality $y < 2x - \frac{1}{4}$ which has a strict inequality. Shade the lower half of the line(Fig 4.9)

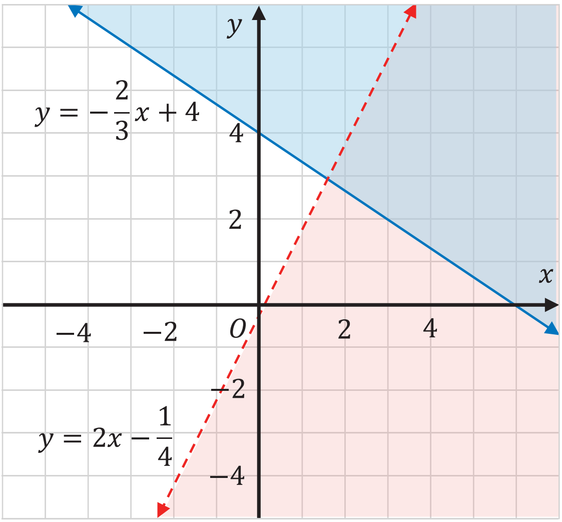


Figure 4.9

Draw a dashed vertical line $x = 4$, which is the related equation of the third inequality. Since $x < 4$, shade the left half of the line (Fig 4.10).

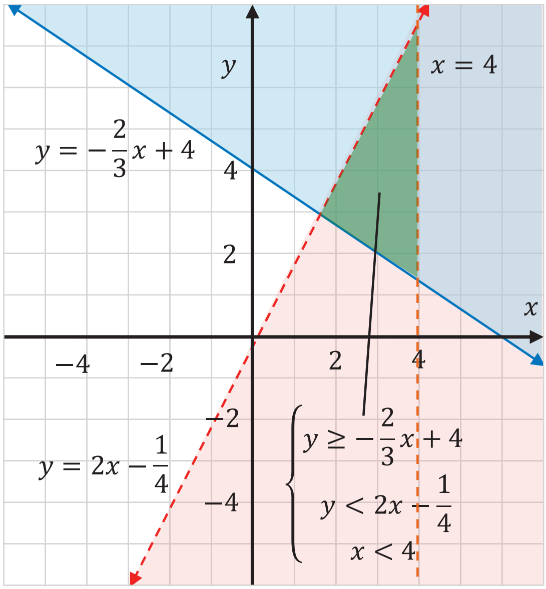


Figure 4.10

The solution of the system of inequalities is the intersection region of the solutions of the three inequalities.

Exercise 4.5

Solve the following systems of linear inequalities by using graphical method.

a. $\begin{cases} y > 3x - 4 \\ y \leq -2x \end{cases}$

b. $\begin{cases} y \geq -3x - 6 \\ y > 4x - 4 \end{cases}$

c. $\begin{cases} y \leq \frac{1}{2}x + 2 \\ y \leq -\frac{2}{3}x + 1 \end{cases}$

d. $\begin{cases} y < -\frac{3}{5}x + 4 \\ y \leq \frac{1}{3}x + 3 \end{cases}$

e. $\begin{cases} y < -\frac{3}{7}x - 1 \\ y > \frac{4}{5}x + 1 \end{cases}$

f. $\begin{cases} 2x + y < 4 \\ x - y \leq 4 \\ x > 2 \end{cases}$

4.3 Inequalities Involving Absolute Value

Graphing absolute value inequalities

Activity 4.4

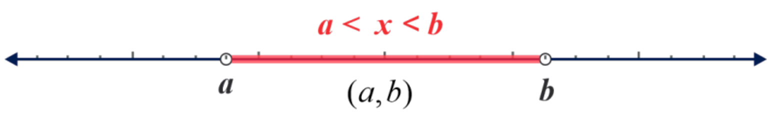
Which of the absolute value inequalities represent the number line indicated by red color below?

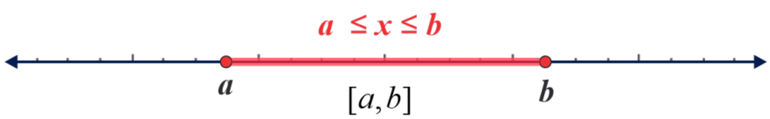
a. $|x + 4| > 3$ b. $|x + 4| \geq 3$ c. $|x + 4| < 3$ d. $|x + 4| \leq 3$

While graphing absolute value inequalities on a number line, you have to keep the following things in mind.

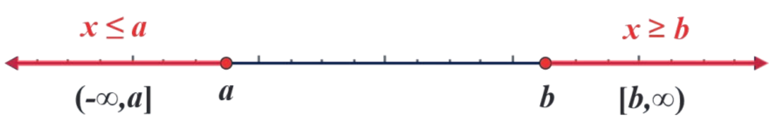
- Use open dots at the endpoints of the open interval (a, b).
- Use closed dots at the endpoints of the closed interval $[a, b]$.

Inequalities with < or ≤





Inequalities with <, > or ≤, ≥



Example

Graph the following absolute value inequality on a number line.

a. $|x| \leq 1$ **b.** $|x| > 1$

Solution:

a. $|x| \leq 1$ are the numbers that satisfy $-1 < x < 1$.

b. $|x| > 1$ are the numbers that satisfy $x < -1$ or $x > 1$.

Exercise 4.6

Put the following absolute value inequalities on a number line.

a. $|x| < 2$ b. $|x| \leq 3$ c. $|x| \leq 5$

d. $|x| > 4$ e. $|x| > \frac{3}{2}$

Inequalities involving absolute value

Assume k is an algebraic expression and c is a positive number.

The solutions of $|k| < c$ are the numbers that satisfy $-c < k < c$

The solutions of $|k| > c$ are the numbers that satisfy $k > c$ or $k < -c$. These rules are valid if $<$ is replaced by $\leq$ and $>$ is replaced by $\geq$.

Example 1

Solve the absolute value inequality $|x + 2| < 4$.

Solution:

The solutions of $|k| < c$ are the numbers that satisfy $-c < k < c$.

Hence, $|x + 2| < 4$

$$-4 < x + 2 < 4$$

$$-4 < x + 2 \ \textbf{and} \ x + 2 < 4$$

$$-4 < x + 2$$

$$-6 < x$$

$$x > -6$$

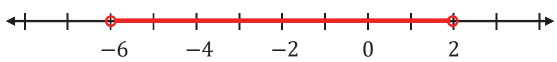
On the other hand, $x + 2 < 4$

$$x < 2.$$

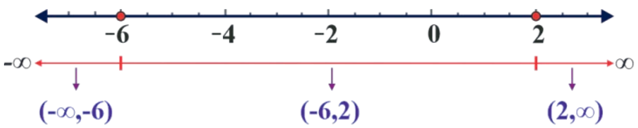
Therefore, the solution is $x > -6$ and $x < 2$ or equivalently, $-6 < x < 2$

The solution using interval notation is $(-6, 2)$

The solution using a number line is shown as follows:

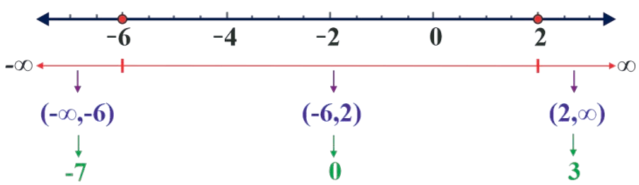


Note: Here, we can see that the number line is divided into 3 parts.



Take a random number from each of these intervals and substitute it in the given inequality.

Identify which of these numbers satisfy the given inequality. For instance, $x = 0$ does satisfy the inequality, while neither $x = -7$ nor $x = 3$ satisfy it.



Therefore, the solution of the given inequality is, $(-6, 2)$ or $-6 < x < 2$

Alternatively, one can solve $|x + 2| < 4$ using $|k| < c$ if and only if $-c < k < c$

$$-4 < x + 2 < 4$$

$$-4 - 2 < x < 4 - 2$$

$$-6 < x < 2$$

We also note the following:

i. If the problem was $|x+2| \leq 4$, then the solution would have been $[-6, 2]$ or $-6 \leq x \leq 2$

If $|x+2| > 4$, then the solution would have been $(-\infty, -6) \cup (2, \infty)$ or $x < -6$ or $x > 2$.

ii. Also If $|x+2| \geq 4$, then the solution would have been $(-\infty, -6] \cup [2, \infty)$ or $x \leq -6$ or $x \geq 2$.

Example 2

Solve the absolute value inequality $|3x - 2| \geq 7$.

Solution:

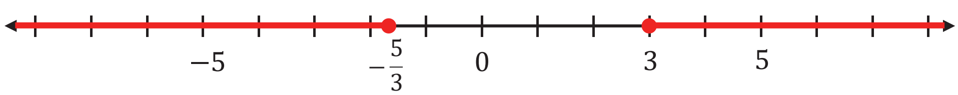
The solutions of $|k| \geq c$ are the numbers that satisfy $k \geq c$ **or** $k \leq -c$.

Hence, $|3x - 2| \geq 7$

$3x - 2 \geq 7$ **or** $3x - 2 \leq -7$

$3x \geq 9$ **or** $3x \leq -5$

$x \geq 3$ **or** $x \leq -\frac{5}{3}$. (Represent on a number line)



Steps for Solving Linear Absolute Value Inequalities:

1. Identify what the absolute value inequality is set "equal" to…

a. If the absolute value is less than zero, there is no solution.

b. If the absolute value is less than or equal to zero, there is one solution. Just set the argument equal to zero and solve.

c. If the absolute value is greater than or equal to zero, the solution is all real numbers.

d. If the absolute value is greater than zero, the solution is all real numbers except for the value which makes it equal to zero. This will be written as a union.

2. Graph the answer on a number line and write the answer in interval notation.

Example 3

Solve the absolute value inequality $|-x+4| < 0$.

Solution:

As said above in step (a), there is no solution as the absolute value is less than 0. We can prove it as follows. Using $|k| < c$, if and only if $-c < k < c$, the inequality becomes, $-0 < -x+4 < 0$

$$0 < -x+4 < 0$$

$$0 < -x+4 \quad \text{and} \quad -x+4 < 0$$

This leads to $x < 4$ and $x > 4$. Hence, there is no real number satisfies x. Thus, this inequality has no solution.

Example 4

Solve the absolute value inequality $|-5x-7| \leq 0$.

Solution:

As said above in steps (b.), there is one solution when the given inequality is equal to zero. Thus, the solution of $|-5x-7| = 0$ is the solution of $|-5x-7| \leq 0$.

$$|-5x-7| = 0$$

$$-5x-7 = 0$$

$$x = -\frac{7}{5}$$

Example 5

Solve the absolute value inequality $|6x+8| > -2$.

Solution:

As absolute value of a number is always greater than a negative number, the value of x is all real numbers.

Exercise 4.7

Solve the following absolute value inequalities.

a. $|x+3| < 7$ b. $|x-5| \leq 2$ c. $|x-7| > 4$

d. $|2x - 3| \geq 5$ e. $|x + 8| < 0$ f. $|3 - x| \leq 0$

g. $|4x - 5| \geq -2$

4.4 Quadratic Inequalities

In unit 3 section 3.3.3, you have learned how to solve quadratic equations using factorization, completing the square method or quadratic formula.

Definition 4.2

The general forms of the quadratic inequalities are: $ax^2 + bx + c < 0$, $ax^2 + bx + c \leq 0$, $ax^2 + bx + c > 0$ and $ax^2 + bx + c \geq 0$ where $\neq a$ and a, b, c $\in \mathbb{R}$

For instance, inequalities such as $x^2 - 6x - 16 \geq 0$, $2x^2 - 11x + 12 > 0$, $x^2 + 4 > 0$, $x^2 - 3x + 2 \leq 0$ are quadratic inequalities.

4.4.1 Solving quadratic inequalities using product properties

Activity 4.5

1. a) If the product of two real numbers is greater than zero, what can you say about the numbers?

   b) If the product of two real numbers is less than zero, what can you say about the numbers?

   c) If the product of two real numbers is equal to zero, what can you say about the numbers?

2. If possible, factorize each of the following expressions.

   a. $x^2 - 8x + 16$ b. $x^2 - x - 6$

   c. $x^2 + 4x + 7$ d. $3x^2 - 17x + 10$

Product properties:

1. $mn > 0$, if and only if

i. $m > 0$ **and** $n > 0$ or *ii.* $m < 0$ **and** $n < 0$

2. $m \cdot n < 0$, if and only if

i. $m > 0$ **and** $n < 0$ or *ii.* $m < 0$ **and** $n > 0$

Example 1

Solve $(x-3)(x-1) < 0$

Solution:

Using the product property 2, the first case is

$$x - 3 > 0 \text{ and } x - 1 < 0$$

This leads to $x > 3$ and $x < 1$, which is not possible.

The second case is $x - 3 < 0$ and $x - 1 > 0$

This leads to $x < 3$ and $x > 1$, which is $1 < x < 3$.

Therefore, the solution is $1 < x < 3$, or $(1, 3)$ in interval notation.

Example 2

Solve the inequality $(x-4)(x-2) > 0$

Solution:

Using the product property,

Case i: $x - 4 > 0$ and $x - 2 > 0$

$x > 4$ and $x > 2$

$x > 4$ or

Case ii: $x - 4 < 0$ and $x - 2 < 0$

$x < 4$ and $x < 2$

Thus, $x < 2$

Hence, the solution is $x < 2$ or $x > 4$. This is expressed as $(-\infty, 2) \cup (4, \infty)$ in interval notation.

Exercise 4.8

Solve the following quadratic inequalities.

a. $(x-5)(x-4)<0$

b. $(x+1)(x-5)<0$

c. $(x+3)(x-6)\geq 0$

d. $(x-2)(x-7)>0$

e. $(2x-1)(x-3)\leq 0$

f. $(3x+5)(3x+1)>0$

Rearrange the given formula, factorize it, and solve the quadratic inequality.

Example 3

Solve the inequality $x^2-x\geq 12$.

Solution:

$x^2-x\geq 12$ implies $x^2-x-12\geq 0$.

Factorize the quadratic inequality to get $(x-4)(x+3)\geq 0$.

Using the product rule, the product will be greater than or equal to zero if

i. $x-4\geq 0$ and $x+3\geq 0$. Hence, $x\geq 4$ and $x\geq -3$. Thus, $x\geq 4$.

Or

ii. $x-4\leq 0$ and $x+3\leq 0$. Hence, $x\leq 4$ and $x\leq -3$. Thus, $x\leq -3$.

Therefore, the solution of this quadratic inequality is $x\leq -3$ or $x\geq 4$.

Alternatively, the solution is $(-\infty,\ -3]\cup[4,\ \infty)$.

Example 4

Solve an inequality $-x^2+x+12\leq 0$

Solution:

$-x^2+x+12\leq 0$ implies $x^2-x-12\geq 0$

The rest of the solution is the same as Example 3 above.

Therefore, the solution of this quadratic inequality is $x\leq -3$ or $x\geq 4$.

Alternatively, the solution is $(-\infty,\ -3]\cup[4,\ \infty)$.

Exercise 4.9

Solve the following quadratic inequalities using product rule.

a. $x^2 - 6x + \quad < 0$

b. $x^2 + 6x + \quad \geq 0$

c. $x^2 - 2x - \quad > 0$

d. $x^2 - 7x \leq -6$

e. $-x^2 + 4 < 0$

f. $2x^2 + x - 15 \leq 0$

g. $-2x^2 + 5x + 12 \geq 0$

4.4.2 Solving quadratic inequalities using sign chart

Activity 4.6

Fill in the table as +++, − − − or 0 as indicated in the table.

Factors	$x < -1$	$x = -1$	$-1 < x < 2$	$x = 2$	$x > 2$
$x + 1$	$- - -$				
$x - 2$		$-$	$- - -$	0	
$f(x) = (x + 1)(x - 2)$					$+ + +$

Sign chart is a table or number line used to solve inequalities which can be factorized into linear binomials. For example, $(ax + b)(cx + d) > 0$. It could also be less than or equal or greater than or equal.

Steps in solving quadratic inequalities by sign chart:

1. Re-write the inequality to get a 0 on the right-hand side.
2. Factor (if possible) the left-hand side.
3. "Graph" each factor on a sign chart using a display of signs. A root of any factor is a key point.
4. Each key point or interval is displayed using a column.

Example 1

Solve $(x-4)(x+3)<0$

Solution:

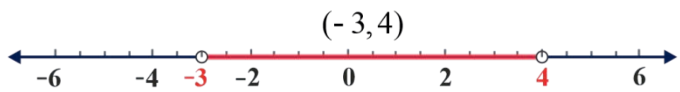
The factors of the left-hand side is $x=-3$ and $x=4$. Using the factors, we prepare the sign chart as follows:

Factors	$x<-3$	$x=-3$	$-3<x<4$	$x=4$	$x>4$
$x-4$	$- - -$	$-$	$- - -$	0	$+ + +$
$x+3$	$- - -$	0	$+ + +$	$+$	$+ + +$
$(x-4)(x+3)$	$+ + +$	0	$- - -$	0	$+ + +$

From the table above, we see that $(x-4)(x+3)<0$ if $-3<x<4$.

Alternatively, $x \in (-3,4)$

We represent this on a number line as follows:



Example 2

Solve $(x+3)(x+2) \geq 0$

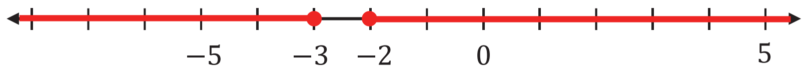
Solution:

Factors	$x<-3$	$x=-3$	$-3<x<-2$	$x=-2$	$x>-2$
$x+3$	$- - -$	0	$+ + +$	$+$	$+ + +$
$x+2$	$- - -$	$-$	$- - -$	0	$+ + +$
$(x+3)(x+2)$	$+ + +$	0	$- - -$	0	$+ + +$

From the table above, we see that $(x+3)(x+2) \geq 0$ when $x \leq -3$ *or* $x \geq -2$.

The solution can be written using interval notation as $(-\infty,-3] \cup [-2, \infty)$

We represent this on a number line as follows:



Exercise 4.10

Use sign chart to solve the following quadratic inequalities. Check the solution with the product property method.

a. $(x-3)(x+1)<0$

b. $(x-3)(x+2)>0$

c. $(2x-1)(3x+4)>0$

d. $x^2-x-12\leq 0$

e. $x^2+2x-15>0$

f. $5-4x-x^2>0$

g. $1-x-2x^2<0$

h. $10x^2-19x+6\leq 0$

4.5 Some Applications of Inequalities

Activity 4.8

Write each statement as linear inequality in two variables.

a. The sum of the costs of 10 books and 4 pens is greater than Birr 180.

b. Given that Ariat is taller than Medina. The difference between the height of Medina and Ariat is at least 10cm.

c. Five times the length of a ruler decreased by 5cm is less than the height of Musa.

d. In a month, the total amount the family spends for food and educational expenses and recreation is at most Birr 5,000.

Activity 4.9

The total amount Meskerem paid for 5 kilos of rice and 2 kilos of coffee is less than Birr 600.

a. What mathematical statement represents the total amount Meskerem paid? Define the variables used.

b. Suppose a kilo of rice costs Birr 35. What could be the greatest cost of a kilo of coffee to the nearest birr?

c. Suppose Meskerem paid more than Birr 600 and each kilo of rice costs Birr 34. What could be the least amount she will pay for 2 kilos of coffee? Of coffee to the nearest birr?

Example 1

Kebede has started a saving account for buying a tractor. He saved Birr 6,000 up to last month and plans to add Birr 1,000 each month until he saves more than Birr 80,000. Write and solve the inequality. Then, interpret the answer. How many months does he need to save more than Birr 80,000?

Solution:

More than Birr 80,000 means greater than Birr 80,000. *Let* "m" be the number of months he saves.

$$6{,}000 + 1{,}000m > 80{,}000$$

$$1{,}000m > 74{,}000$$

$$m > 74$$

Since the number of months is greater than 74, it will take him more than 6 years 2 months to save enough money to buy a tractor.

Example 2

Chaltu likes to text messages to her friends using her cell phone. She is charged Birr 0.20 each time she types a message. Her telephone has Birr 50.00 in her account. She is allowed to have a bill that is at most Birr 80.00. Write and solve the inequality. Then, interpret the answer. How many messages can she text message at most?

Solution:

At most means less than or equal to and then we have

$$0.2m + 50 \leq 80$$

$$0.2m \leq 30$$

$m \leq 150$

Since the number of messages is less than or equal to 150, she can send a maximum of 150 text messages.

Example 3

The length of a normal human pregnancy is from 37 to 41 weeks, inclusive.

a. Write an inequality representing the normal length of a pregnancy.

b. Write a compound inequality representing an abnormal length for a pregnancy.

Solution:

Let w be the number of weeks that a human pregnancy takes, then

a. The inequality representing the normal length of a pregnancy will be $37 \leq w \leq 41$

b. A compound inequality representing an abnormal length for a pregnancy will be $w < 37$ or $w > 41$

Example 4

Dendir works two part time jobs in order to earn enough money to meet his obligations of at least Birr 240 a week. His job in food service pays Birr 10 per hour and his tutoring job on campus pays Birr 15 per hour. How many hours does Dendir need to work at each job to earn at least Birr 240?

a. Let x be the number of hours Dendir works at the job in food service and let y be the number of hours he works tutoring. Write an inequality that would model this situation.

b. Graph the inequality.

c. Find three ordered pairs (x, y) that would be solutions to the inequality. Then, explain what that means for Dendir.

Solution:

a. Dendir earns Birr 10 per hour at the job in food service and Birr 15 per hour tutoring. At each job, the number of hours multiplied by the hourly wage will give the amount earned at that job.

Amount earned at the food service job plus the amount earned tutoring is at least Birr 240.

$10x + 15y \geq 240.$

b. To graph the inequality (Fig 4.11), we put in slope-intercept form.

$10x + 15y \geq 240$

$15y \geq -10x + 240$

$y \geq -\frac{2}{3}x + 16$

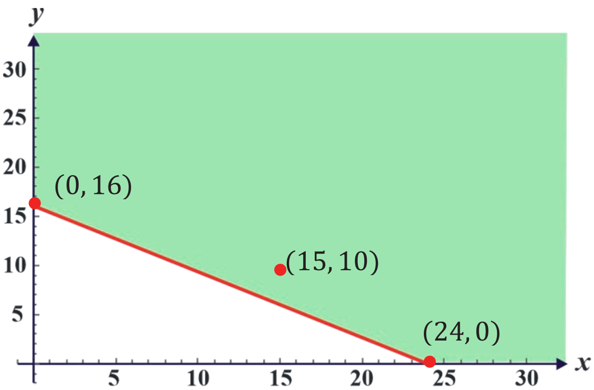


Figure 4.11

c. From the graph, we see that the ordered pairs represent three of the infinitely many solutions.

Check the values in the inequality, $10x + 15y \geq 240$.

$(15, 10)$	$(0, 16)$	$(24, 0)$
$(LHS) = 10(15) + 15(10)$ $= 300$ $(LHS) > (RHS)$, true.	$(LHS) = 10(0) + 15(16)$ $= 240$ $(LHS) = (RHS)$, true.	$(LHS) = 10(24) + 15(0)$ $= 240$ $(LHS) = (RHS)$, true.
This means that Dendir works for food service for 15 hours and for tutoring for 10 hours per week respectively. Then, he earns Birr 300 per week.	This means that Dendir works for tutoring for 16 hours per week. Then, he earns Birr 240 per week.	This means that Dendir works for food service for 24 hours per week. Then he would earns Birr 240 per week.

Exercise 4.11

1. Answer the following questions.
   a. The difference of a number and 6 is between 0 and 8. Find all such numbers.
   b. Twice a number is between 2 and 12. Find all such numbers.
   c. One plus twice a number is greater than 1and less than 5. Find all such numbers.
   d. One-third of a number is either less than 2 or greater than 5. Find all the natural numbers.
2. On Addis Ababa's interstate highway, the speed limit is 55 km/h. The minimum speed limit is 45 km/h . Write a compound inequality that represents the allowable speeds.
3. The normal number of white blood cells for human blood is between 4800 and 10,800 cells per cubic millimeter, inclusive.
   a. Write an inequality representing the normal range of white blood cells per cubic millimeter.
   b. Write a compound inequality representing abnormal levels of white blood cells per cubic millimeter.
4. The ideal diameter of a piston for one type of car is 88 mm. The actual diameter can vary from the ideal by at most 0.007 mm. Find the range of acceptable diameter for the piston.
5. Assume that you are allowed to go within 9 km/h of the speed limit of 65 km/h without getting a ticket. Write an absolute value inequality that models this situation. Write an absolute inequality that shows the speed you are allowed to go without getting a ticket.
6. One leg of a right triangle is 3 cm longer than the other. How long should the shorter leg be to ensure the area is at least 14 cm^2?

7. Fatima, who lives in Jimma zone, needs to make a rectangular coffee plot which has an area of at most $18\ m^2$. The length should be 3 m longer than the width. What are the possible dimensions of the plot?
8. Almaz is selling sambusa at a school. She sells two sizes, small (which has 1 scoop of beans inside) and large (which has 2 scoops of beans). She knows that she can get a maximum of 70 scoops of beans out of her supply. She charges Birr 3 for a small sambusa and Birr 5 for a large one. Almaz wants to earn at least Birr 120 to give back to the school. Let x be the number of small sambusa, and y be the number of large one? Write and graph a system of inequalities that models this situation.

Summary

1. The open interval (a, b) with end-points a and b is the set of all real numbers x such that $a < x < b$.
2. The closed interval $[a, b]$ with end-points a and b is the set of all real numbers x such that $a \leq x \leq b$.
3. The half-open interval or half-closed interval $(a, b]$ with end points a and b is the set of all real numbers x such that $a < x \leq b$.
4. The half-open interval or half-closed interval $[a, b)$ with end points a and b is the set of all real numbers x such that $a \leq x < b$.
5. For any positive real number a, the solution set of:
   - **i.** the inequality $|x| < a$ is $-a < x < a$ and
   - **ii.** the inequality $|x| > a$ is $x < -a$ or $x > a$.
6. Linear inequalities in two variables represent the inequality between two algebraic expressions involving the two variables.
7. An inequality that can be reduced to either $ax^2 + bx + c \leq 0$, $ax^2 + bx + c < 0, ax^2 + bx + c \geq 0$ or $ax^2 + bx + c > 0$ where a, b and c are constants and $a \neq 0$ is called a **quadratic inequality.**
8. A quadratic inequality can be solved by using product properties, sign chart or graphically.
9. Property of product rule.
   - **a.** If $m \cdot n > 0$, then $m > 0$ and $n > 0$ or $m < 0$ and $n < 0$.
   - **b.** If $m \cdot n < 0$, then $m > 0$ and $n < 0$ or $m < 0$ and $n > 0$.

Review Exercise

1. If $x > y$, what can you say about the inequality sign if we multiply both side of the inequality by a negative number? By a positive number? By zero?

2. If x is a natural number, then the solution set for the inequality $3x + 7 < 16$ is

a. $\{1,2\}$ **b.** $\{2,3\}$ **c.** $\{\ldots..0, 1, 2, 3\}$ **d.** $\{4, 5, 6, \ldots..\}$

3. Set up an algebraic inequality for

a. The sum of 7 and three times a number is less than or equal to 1.

b. If three is subtracted from two times a number, then the result is greater than or equal to nine.

4. Solve the following inequalities.

a. $\frac{2}{3}\left[x - \left(1 - \frac{x-2}{3}\right) + 1\right] \le x$ **b.** $2 - [-2(x+1)] - \frac{x-3}{2} \le \frac{2}{3}x - \frac{5x-3}{12} + 3x$

c. $\frac{3x+1}{7} - \frac{2-4x}{3} \ge \frac{-5x-4}{14} + \frac{7}{6}x$

5. Solve the following simultaneous inequalities.

a. $\begin{cases} x + 7 > 4 \\ x + 2 < 5 \end{cases}$ **b.** $\begin{cases} 3x + 2 > 7 \\ x + 1 < 5 \end{cases}$ **c.** $\begin{cases} 2x + 1 \ge 9 \\ 3x + 4 < 25 \end{cases}$

6. Solve the following simultaneous inequalities:

a. $\begin{cases} -2x + y > -4 \\ 3x - 6y \ge 6 \end{cases}$ **b.** $\begin{cases} -3x + 2y > 6 \\ 6x - 4y > 8 \end{cases}$ **c.** $\begin{cases} y \ge -4 \\ y < x + 3 \\ y \le -3x + 3 \end{cases}$

7. Solve the following inequalities:

a. $|2x - 1| - 7 \le -5$ **b.** $|x - 1| \ge -3$

c. $|x - 1| \le -3$ **d.** $\left|\frac{2x-9}{3}\right| < 4$

e. $|6 - 2x| \le 2$

8. Solve the following quadratic inequalities using product rule, sign chart and graphically:

a. $(x - 1)(x - 2) > 0$ **b.** $(x + 3)(x - 5) \le 0$

c. $x^2 + 2x - 3 \ge 0$ **d.** $x^2 + 6x > -2x$

e. $4x - x^2 > 12$ **f.** $x^2 + 8x + 16 \ge 0$

g. $x^2 - 4x - 12 \leq 0$ **h.** $4x^2 > 12x$

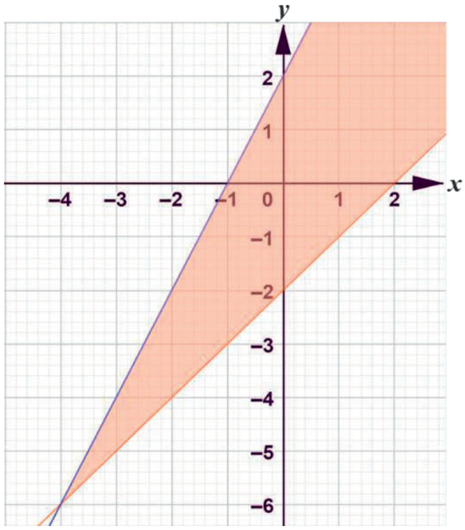
9. Which of the following is the solution for the inequality $|3x + 7| \leq 5$?

A. $\frac{2}{3} \leq x \leq 4$ B. $x \leq -\frac{2}{3}$ C. $x \leq -4$ D. $-4 \leq x \leq -\frac{2}{3}$

10. Which of the following point is a solution to the inequality $\begin{cases} x - 2y > -1 \\ 3x - y < -3 \end{cases}$

A. $(2, 1)$ B. $(-2, -1)$ C. $(2, -1)$ D. $(-2, 1)$

11. Which of the following system of inequalities represent the following graph?



A. $\begin{cases} y > x - 2 \\ y \leq 2x + 2 \end{cases}$ B. $\begin{cases} y \geq x - 2 \\ y \leq 2x + 2 \end{cases}$ C. $\begin{cases} y > x - 2 \\ y > 2x + 2 \end{cases}$ D. $\begin{cases} y \geq x - 2 \\ y < 2x + 2 \end{cases}$

12. The solution for the quadratic inequality $x^2 - x > 12$ is:

A. $x > -3$ or $x > 4$ B. $x < -3$ or $x > 4$

C. $-3 < x < -4$ D. $x > -3$ and $x < 4$

13. A technician measures an electric current which 0.036A with a possible error of± 0.002 A. Write this current, i, as an inequality with absolute values.

14. A shop owner has determined that the relationship between monthly rent charged for store space r (in hundred birr per square meter) and monthly profit $P(r)$ (in thousands of birr) can be approximated by the function $P(r) = -9r^2 + 8r + 1$. Solve each quadratic equation or inequality. Explain what each answer tells about the relationship between monthly rent and profit for the shop owner?

UNIT 5

INTRODUCTION TO TRIGONOMETRY

Unit Outcomes

By the end of this unit, you will be able to:

- **Identify the hypotenuse, opposite and adjacent of a right-angled triangle.**
- **Describe the basic properties of a right-angled triangle.**
- **Define sine, cosine, and tangent ratio.**
- **Find trigonometric values of angles from trigonometric table.**

Unit Contents

KEY Words

- right-angled triangle
- trigonometric table
- trigonometric ratio (sine, cosine and tangent)
- Pythagoras theorem

INTRODUCTION

You have already studied about triangles, in particular, right-angled triangles, in your earlier classes. In this unit, you will learn the relationship between angles and sides of a right-angled triangle.

5.1 Revision on Right-angled Triangles

Definition 5.1 Right-angled triangle

A right-angled triangle is a type of triangle that has one of its angles equal to 90 degrees.

1. The opposite side to the right-angle is called hypotenuse.
2. The relationship among the three sides is: $b^2 + p^2 = h^2$.

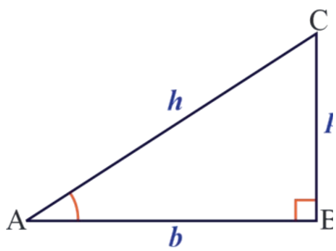


Figure 5.1

Example 1

a. What is the sum of the maximum degree measure of the two interior angles other than the right angle?

b. If one of the angles of a triangle is 90°and the other two are equal to 45°each, what do you call such a triangle?

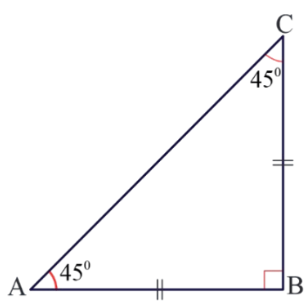


Figure 5.2

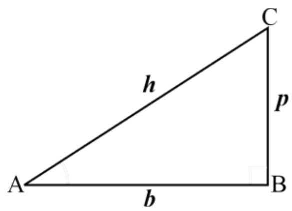
Solution:

a. As you observed in figure 5.1, one of the angles of a right-angled triangle is 90°. The sum of the measure of the other two angles of this triangle is 90°. So the sum of the maximum degree measure of the two other interior angles of a right-angled triangle is 90°.

b. If one of the angles of a triangle is 90° and the other two angles measure 45° each, then the triangle is called an isosceles right-angled triangle, where the adjacent sides to 90° are equal in length (figure 5.2).

Note

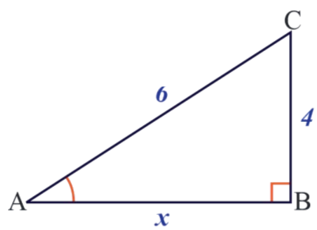
For right-angled triangle, we also have the following basic properties:



- One angle is always 90° or right angle.
- The opposite side to angle 90° is hypotenuse.
- The hypotenuse is always the longest side.
- The sum of the other two interior angles is equal to 90° (each angle is acute).
- The other two sides adjacent to the right angle are called base and perpendicular side. (b is the base and p is the perpendicular side).
- If one of the angles is 90° and the other two angles are equal to 45° each, then the triangle is called an isosceles right-angled triangle, where the adjacent sides to 90° are equal in length.
- If b, p and h are sides of a right-angled triangle as shown in the above figure, we can write a relation using the **Pythagoras theorem,** that is $b^2 + p^2 = h^2$.

Example 2

Find the length of the missing side.



Solution:

Using the Pythagoras theorem, $x^2 + 4^2 = 6^2$.

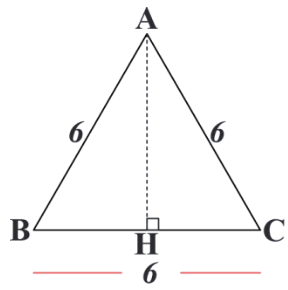
It results in $x^2 = 20$. Since $x > 0$, $x = 2\sqrt{5}$.

Example 3

Find the height of an equilateral triangle ABC with length of 6units.

Solution:

Consider equilateral triangle as shown at the right. When drawing a line segment from A and perpendicular to side BC, H is midpoint of BC. BH = 3unit. Using Pythagoras theorem, let AH = x ($x > 0$), then $BH^2 + AH^2 = AB^2$



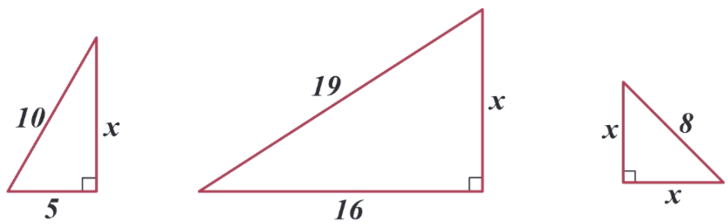
$$3^2 + x^2 = 6^2$$

$$x^2 = 27$$

This results in $AH = x = 3\sqrt{3}$ units.

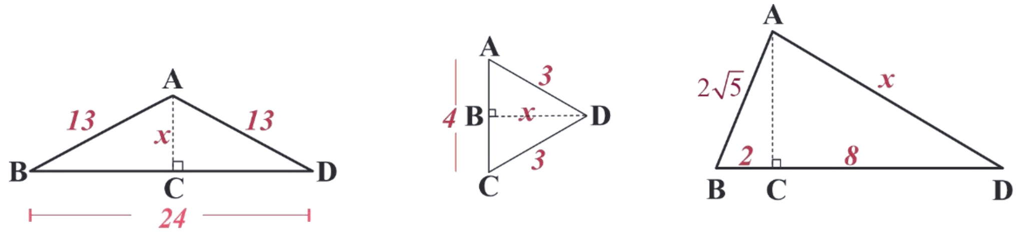
Exercise 5.1

1. Find the value of x. Express your answer in simplest radical form.



(a) (b) (c)

2. Find the value of x. If your answer is not an integer give it in the simplest radical form.

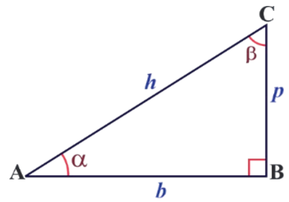


(a) (b) (c)

Conversion of the Pythagoras theorem

Conversion of the Pythagoras theorem

If $b^2 + p^2 = h^2$, then $\angle B = 90°$.



Example 1

Which one of the following can be sides of a right-angled triangle (the sides are with the same unit of length)?

a. $5, 6, 2$ **b.** $3, 4, 5$ **c.** $\sqrt{5}, 2, 3$

Solution:

a. First identify the longest side (hypotenuse). From the property of right-angled triangle, the longest side is the hypotenuse and the other would be the base and perpendicular side.

So, we can take 2 and 5 as length of the base and perpendicular side, respectively, and 6 as the hypotenuse, as shown in the figure. Now, check if Pythagoras theorem is satisfied or not, that is $b^2 + p^2 = h^2$ or not. The left hand side (LHS) of the equation is $5^2 + 2^2 = 29$ and the right hand side (RHS) of the equation is $6^2 = 36$. Here, LHS ≠ RHS. Hence, the triangle is not a right-angled triangle.

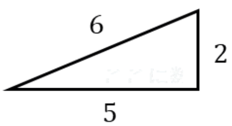


Figure 5.3

b. Consider $3, 4$ and 5 as a, b and c respectively. Check whether these numbers satisfy the theorem $b^2 + p^2 = h^2$.

LHS $= 3^2 + 4^2 = 25$ and RHS $= 5^2 = 25$. Thus, LHS = RHS . Therefore, it is a right-angled triangle.

c. As $2^2 < \left(\sqrt{5}\right)^2 < 3^2$, then $2 < \sqrt{5} < 3$. Hence, $h = 3$. Using equation

$b^2 + p^2 = h^2$,LHS $= \left(\sqrt{5}\right)^2 + 2^2 = 9$ and RHS $=3^2 = 9$. Thus LHS = RHS.

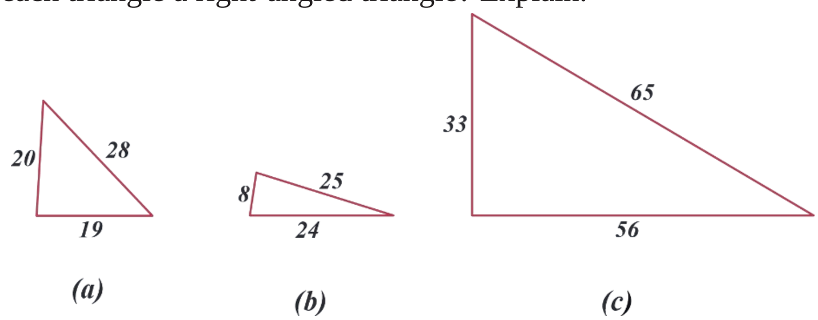
Therefore, it is a right-angled triangle.

Exercise 5.2

1. Identify whether the given sides can form a right-angled triangle or not provided that the units for each length is identical.

   a. $4, 6, 8$ b. $\sqrt{2},\ \sqrt{3},\ \sqrt{5}$ c. $1, 2, \sqrt{3}$ d. $\sqrt{6},\ 3,\ \sqrt{3}$

2. Is each triangle a right-angled triangle? Explain.



5.2 Trigonometric Ratios

In this subsection, you will learn some ratios of the sides of a right-angled triangle with respect to its angles, called trigonometric ratios of the angle. We will restrict our discussion to acute angles only. However, these ratios can also be extended to other angles.

Could you think some examples from our surroundings where right-angled triangles can be imagined?

- Suppose 9th grade students of your school are visiting Addis Ababa. Now, if a student is looking at the top of the building (head office of the Commercial Bank of Ethiopia)

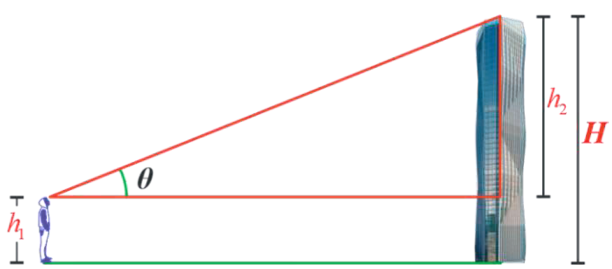


Figure 5.4

a right-angled triangle can be imagined, as shown in fig 5.4. Can you find out the height of the building without actually measuring it?

In the above situation, the height can be found by using some mathematical techniques, which come under a branch of mathematics called ‘trigonometry’. Trigonometry is derived from two Greek words 'trigonon' which means 'triangle' and 'metron' meaning 'measure'. Put together, the words mean "triangle measuring". In fact, trigonometry is the study of relationships between the sides and angles of a triangle. The earliest known work on trigonometry was recorded in Egypt and Babylon. Early astronomers used it to find out the distances of the stars and planets from the Earth. Even today, most of the technologically advanced methods used in Engineering and Physical Sciences are based on trigonometric concepts.

Now let us define the trigonometric ratio.

Definition 5.2 Trigonometric ratios

The trigonometric ratios of a given angle are defined by the ratios of two sides of a right-angled triangle. These trigonometric ratios remain unchanged as long as the angle remains the same, that is, they are independent of the size of the triangle provided the angle remains the same.

There are six trigonometric ratios. But here we will define only the first three trigonometric ratios as follows.

Consider the following right-angled triangle as shown in the figure 5.5 below.

For angle A, the trigonometric ratios sine, cosine and tangent are defined as

$$\text{sine of } \angle A = \frac{\text{side opposite to angle } A}{\text{hypotenuse}} = \frac{BC}{AC}$$

$$\text{cosine of } \angle A = \frac{\text{side adjacent to angle } A}{\text{hypotenuse}} = \frac{AB}{AC}$$

$$\text{tangent of } \angle A = \frac{\text{side opposite to angle } A}{\text{side adjacent to angle } A} = \frac{BC}{AB}$$

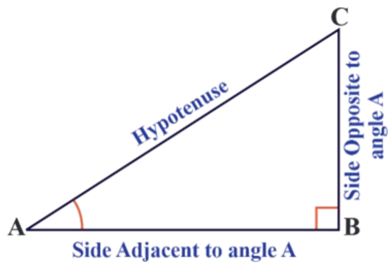


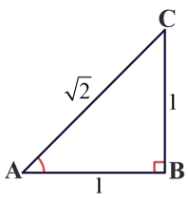
Figure 5.5

The above trigonometric ratios ***sineA***, ***cosine A*** and ***tangent A*** are abbreviated as ***sin A***, ***cos A*** and ***tan A,*** respectively.

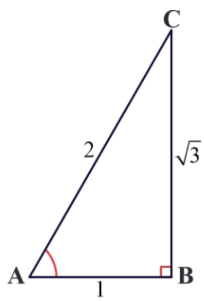
So, the trigonometric ratios of an acute angle in a right-angled triangle express the relationship between the angle and the length of its sides.

Example

Find the trigonometric ratios (sine, cosine and tangent) for angle A in the following figures.



(a)



(b)

Solution:

a. Using definition 5.2, $\sin A = \frac{Opposite}{Hypotenuse} = \frac{1}{\sqrt{2}}$, $\cos A = \frac{adjacent}{hypotenuse} = \frac{1}{\sqrt{2}}$ and

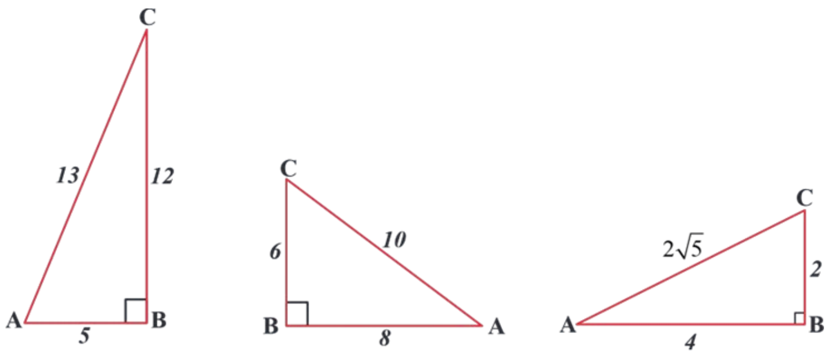
$\tan A = \frac{opposite}{adjacent} = \frac{1}{1} = 1.$

b. Similar to (a) above, $\sin A = \frac{Opposite}{Hypotenuse} = \frac{\sqrt{3}}{2}$, $\cos A = \frac{adjacent}{hypotenuse} = \frac{1}{2}$ and

$\tan A = \frac{opposite}{adjacent} = \frac{\sqrt{3}}{1} = \sqrt{3}.$

Exercise 5.3

1. Find the trigonometric ratios sine, cosine, and tangent of the angle A for each of the following right-angled triangles:



(a) (b) (c)

If one of the three trigonometric ratios is given, it is possible to determine the other two trigonometric ratios.

Example

Determine the remaining two trigonometric ratios of an acute angle A of a right-angled triangle if

a. $\sin A = \frac{3}{5}$ **b.** $\tan A = \frac{1}{\sqrt{2}}$

Solution:

a. First draw a right-angled triangle ΔABC as it is given $\sin A = \frac{3}{5}$ which is equal to $\frac{BC}{AC}$. Hence, we have $\sin A = \frac{BC}{AC} = \frac{3}{5}$ and from this we immediately observe the length of side $BC = 3$ units (opposite side) and $AC = 5$ units (hypotenuse). Now, using the Pythagoras Theorem, we have $AB^2 + BC^2 = AC^2$, we determine the side length AB as follows.

$$AB = \sqrt{AC^2 - BC^2}$$
$$= \sqrt{25 - 9}$$
$$= \sqrt{16}$$
$$= 4$$

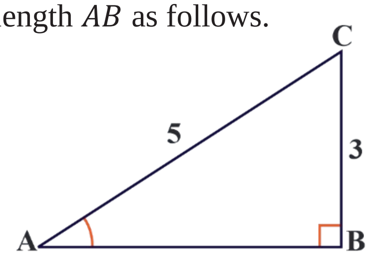


Figure 5.6

Hence, the remaining trigonometric ratios are

$\tan A = \frac{BC}{AB} = \frac{3}{4}$ and $\cos A = \frac{AB}{AC} = \frac{4}{5}$.

b. Following similar procedure as (a), first we need to determine $\cos A$ and $\sin A$ from the given ratio of $\tan A = \frac{1}{\sqrt{2}}$. This is also equal to the ratio $\frac{BC}{AB}$. Hence, $\tan A = \frac{1}{\sqrt{2}} = \frac{BC}{AB}$ and if $BC = 1$, then $AB = \sqrt{2}$. Now, using the Pythagoras Theorem, we determine the length AC as

$$AC = \sqrt{AB^2 + BC^2} = \sqrt{2+1} = \sqrt{3}.$$

Now, the remaining trigonometric ratios are $\cos A = \frac{AB}{AC} = \frac{\sqrt{2}}{\sqrt{3}}$ and $\sin A = \frac{BC}{AC} = \frac{1}{\sqrt{3}}$.

Exercise 5.4

Determine the remaining two trigonometric ratios of an acute angle A of a right-angled triangle if

a. $\sin A = \frac{1}{2}$ b. $\cos A = \frac{2}{3}$ c. $\tan A = \frac{1}{2}$ d. $\sin A = \frac{1}{\sqrt{5}}$

Trigonometric values of basic angles

From lower grade mathematics lessons, you are already familiar with the construction of angles of $0°, 30°, 45°, 60°$ and $90°$. Now you will learn how to find the values of the trigonometric ratios for these angles.

Example 1

Find $\sin 45°, \cos 45°$ and $\tan 45°$.

Solution:

Consider a right-angled triangle ABC right-angled at B.

If one of the acute angles is $45°$, then the other acute angle is also $45°$,

that is $\angle A = \angle C = 45°$(see figure 5.7).

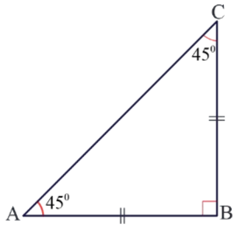


Figure 5.7

Hence, $BC = AB$(because the triangle is isosceles right-angled triangle).

Suppose $BC = AB = a$. Then by Pythagoras Theorem,

$$AC = \sqrt{AB^2 + BC^2} \text{ which implies}$$

$$= \sqrt{a^2 + a^2} = \sqrt{2a^2} = \sqrt{2}a.$$

Using the definition of trigonometric ratios we have:

$$\sin 45° = \frac{\text{side opposite to angle } 45°}{\text{hypotenuse}} = \frac{BC}{AC} = \frac{a}{\sqrt{2}a} = \frac{1}{\sqrt{2}}.$$

$$\cos 45° = \frac{\text{side adjacent to angle } 45°}{\text{hypotenuse}} = \frac{AB}{AC} = \frac{a}{\sqrt{2}a} = \frac{1}{\sqrt{2}}.$$

$$\tan 45° = \frac{\text{side opposite to angle } 45°}{\text{side adjacent to angle } 45°} = \frac{BC}{AB} = \frac{a}{a} = 1.$$

Example 2

Find **i)** $\sin 30°, \cos 30°,$ and $\tan 30°$.

ii) $\sin 60°, \cos 60°,$ and $\tan 60°$.

Solution:

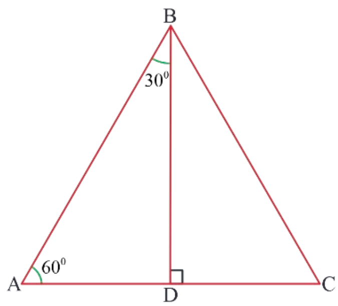


Figure 5.8

Consider half of an equilateral triangle ABC shown as below (Figure 5.8). The triangle ABD is a right-angled triangle with angles 30° and 60° (Fig.5.9). Let the length of side AB be 2. The length of the side AD is half of the length of side AC. Thus $AD = 1$.

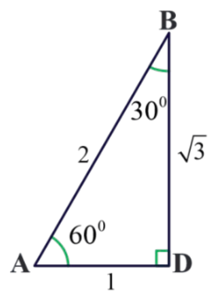


Figure 5.9

By applying the Pythagoras Theorem,

$AD^2 + BD^2 = AB^2$.

Then, $1^2 + BD^2 = 2^2$

$$BD^2 = 4 - 1$$

$$BD^2 = 3$$

$$BD = \sqrt{3} \quad \text{because } BD > 0.$$

Therefore,

$$\sin 30° = \frac{\text{side opposite to angle B}}{\text{hypotenuse}} = \frac{1}{2}. \quad \sin 60° = \frac{\text{side opposite to angle A}}{\text{hypotenuse}} = \frac{\sqrt{3}}{2}.$$

$$\cos 30° = \frac{\text{side adjacent to angle B}}{\text{hypotenuse}} = \frac{\sqrt{3}}{2}. \quad \cos 60° = \frac{\text{side adjacent to angle A}}{\text{hypotenuse}} = \frac{1}{2}.$$

$\tan 30° = \frac{\text{side opposite to angle B}}{\text{side adjacent to angle B}} = \frac{1}{\sqrt{3}}$. $\tan 60° = \frac{\text{side opposite to angle A}}{\text{side adjacent to angle A}} = \frac{\sqrt{3}}{1} = \sqrt{3}$.

Example 3

Find the unknown length of the sides of the following in the right-angled triangle.

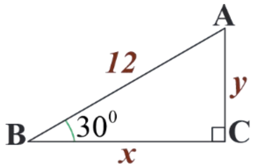


Figure 5.10

Solution:

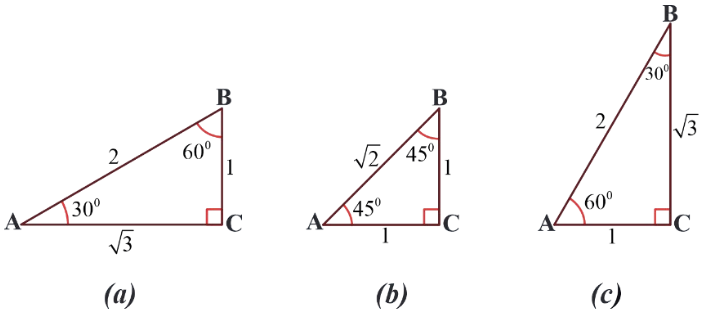
Applying the trigonometric ratios here,

$\cos 30^{\circ} = \frac{x}{12}$, $\cos 30^{\circ} = \frac{\sqrt{3}}{2}$. Then, $\frac{x}{12} = \frac{\sqrt{3}}{2}$, $x = \frac{\sqrt{3}}{2} \times 12 = 6\sqrt{3}$.

Similarly; $\sin 30^{\circ} = \frac{y}{12}$, $\sin 30^{\circ} = \frac{1}{2}$. Then, $\frac{y}{12} = \frac{1}{2}$, $y = 6$.

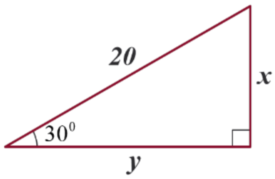
Exercise 5.5

1. Complete the following table column wise using the given triangles (a), (b) and (c) respectively.

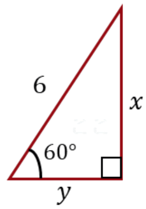


	$\angle A = 30°$	$\angle A = 45°$	$\angle A = 60°$
$\sin A$			
$\cos A$			
$\tan A$			

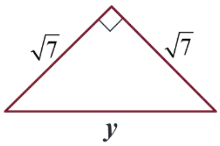
2. Find the unknown values of the following figures. If the value is not an integer, express it in simplest radical form.



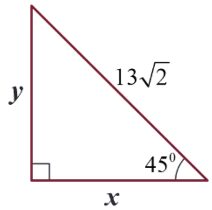
(a)



(b)



(c)



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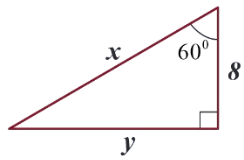

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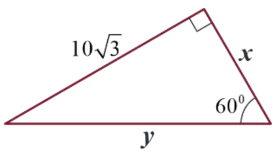

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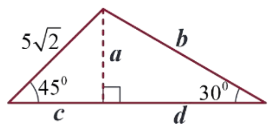

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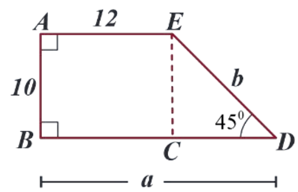


(h)

Trigonometric Ratios of 0° to 90°

We discussed how to determine trigonometric ratio of $30°, 45°$ and $60°$ using examples. To derive the trigonometric ratios of $0°$ and $90°$ needs advanced concept of mathematics and will not be treated at this level. The trigonometric ratios of these two angles together with $30°, 45°$ and $60°$ are indicated in the following table 5.1.

Table 5.1

$\angle A$	$0°$	$30°$	$45°$	$60°$	$90°$
$\sin A$	0	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$	1
$\cos A$	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	0
$\tan A$	0	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$	undefined

Remark: From the above table, you can observe that as $\angle A$ increases from $0°$ to $90°$, $\sin A$ increases from 0 to 1 while $\cos A$ decreases from 1 to 0. On the other hand, the value of $\tan A$ increases from 0, become 1 at 45° and continues to increase until 90°.

Trigonometrical values of angles from a table

$(\sin\theta, \cos\theta \text{ and } \tan\theta \ \ for\ 0° \le \theta \le 90°)$

In the previous discussion we have seen how to determine the trigonometric ratios of $\sin\theta, \cos\theta$ and $\tan\theta$ when θ is a special angle $0°, 30°, 45°, 60°$ and $90°$. Following the same procedure of getting the trigonometric ratio of the given special angles, it is possible to determine the trigonometric ratio of any angle. There are trigonometric tables of approximate values of trigonometric ratios of acute angles that have already been constructed by advanced arithmetical processes. One of such tables is found at the back of this textbook.

Follow the next steps to read the trigonometric values from a trigonometric table:-

- ✓ Identify the trigonometric ratio from the first row if the degree you need is between $0°$ and $45°$ and the last row if the degree is between $45°$and $90°$.
- ✓ Using the column of the trigonometric ratio you need to find and move down or up till you get the required degree.
- ✓ The intersection of the row and column of the above two steps is the trigonometric value of the required degree.

Example

Using a trigonometric table, find the value of each of the following.

a. $\sin 18°$ **b.** $\cos 37°$ **c.** $\tan 70°$

Solution:

a. We can read the trigonometric ratio from a table. Since the given degree is 18, use the first row and take the column of sin then move down till we get

18°. The number we obtain is a value for sin 18°.

That is sin 18° ≈ 0.309017. (You can observe the steps from table 5.2)

Table 5.2

Move Down

Move to the right

	sin	cos	tan
0	0	1	0
1	0.017452	0.999848	0.017455
2	0.03490	0.999391	0.034921
3	0.052336	0.99863	0.052408
4	0.069756	0.997564	0.069927
5	0.087156	0.996195	0.087489
6	0.104528	0.994522	0.10510
7	0.121869	0.992546	0.122784
8	0.139173	0.990268	0.140541
9	0.156434	0.987688	0.158384
10	0.173648	0.984808	0.176327
11	0.190809	0.981627	0.19438
12	0.207912	0.978148	0.212556
13	0.224951	0.97437	0.230868
14	0.241922	0.97030	0.249328
15	0.258819	0.965926	0.267949
16	0.275637	0.961262	0.286745
17	0.292371	0.95630	0.30573
18	0.309017	0.951057	0.324919
19	0.325568	0.945519	0.344327

b. In a similar way, cos 37° ≈ 0.798636.

c. tan 70° ≈ 2.747480

Exercise 5.6

Find the value of each of the following using trigonometric table.

a. sin 23° b. cos 9° c. tan 40°

d. sin 52° e. cos 68° f. tan 82°

Example 1

Given $\cos A = 0.819152$. Using the trigonometric table, find the angle A.

Solution:

On the column corresponding to 'cos', find the given number 0.819152. Corresponding to this number is 35, that is $A = 35°$. (see table 5.3 below).

Table 5.3

Move down till you get cosine value 0.819152

	sin	cos	tan		
0	0	1	0		90
1	0.017452	0.999848	0.017455	57.2900	89
2	0.03490	0.999391	0.034921	28.63628	88
3	0.052336	0.99863	0.052408	19.08115	87
4	0.069756	0.997564	0.069927	14.30068	86
5	0.087156	0.996195	0.087489	11.43006	85
6	0.104528	0.994522	0.10510	9.514373	84
7	0.121869	0.992546	0.122784	8.144353	83
8	0.139173	0.990268	0.140541	7.115376	82
9	0.156434	0.987688	0.158384	6.313757	81
10	0.173648	0.984808	0.176327	5.671287	80
11	0.190809	0.981627	0.19438	5.144558	79
12	0.207912	0.978148	0.212556	4.704634	78
13	0.224951	0.97437	0.230868	4.33148	77
14	0.241922	0.97030	0.249328	4.010784	76
15	0.258819	0.965926	0.267949	3.732054	75
16	0.275637	0.961262	0.286745	3.487418	74
17	0.292371	0.95630	0.30573	3.270856	73
18	0.309017	0.951057	0.324919	3.077686	72
19	0.325568	0.945519	0.344327	2.904214	71
20	0.34202	0.939693	0.36397	2.74748	70
21	0.358368	0.933581	0.383864	2.605091	69
22	0.374606	0.927184	0.404026	2.475089	68
23	0.390731	0.920505	0.424474	2.355855	67
24	0.406736	0.913546	0.445228	2.246039	66
25	0.422618	0.906308	0.466307	2.144509	65
26	0.438371	0.898794	0.487732	2.050306	64
27	0.45399	0.891007	0.509525	1.962612	63
28	0.469471	0.882948	0.531709	1.880728	62
29	0.484809	0.87462	0.554308	1.80405	61
30	0.5000	0.866026	0.57735	1.732053	60
31	0.515038	0.857168	0.60086	1.664281	59
32	0.529919	0.848048	0.624869	1.600336	58
33	0.544639	0.838671	0.649407	1.539867	57
34	0.559192	0.829038	0.674508	1.482563	56
35	0.573576	0.819152	0.700207	1.42815	55
36	0.587785	0.809017	0.726542	1.376383	54

Move to the left till to get the required angle

Example 2

Given $\sin A° = 0.7565$ find the measure of the acute angle A, correct to the nearest degree.

Solution:

In the column of 'sin', search the given number. If we get it directly, take the degree which corresponds to the number. Otherwise, take the approximation to the nearest one. In our case, the given number is not directly available on the table. Two values closest to the given value 0.7565 are 0.75471, the smaller one and 0.766045 the larger one. These values correspond to $49°$ and $50°$, respectively.

The given number is closest to 0.75471(since $|0.7565 - 0.75471| < |0.7565 - 0.766045|$), and it corresponds to $49°$. Therefore, $A = 49°$(to the nearest degree).

Exercise 5.7

1. Find the measure of an acute angle A, correct to the nearest degree if

   a. $\sin A = 0.173648$ b. $\cos A = 0.961262$ c. $\tan A = 0.60086$

   d. $\sin A = 0.798636$ e. $\cos A = 0.438371$ f. $\tan A = 2.355855$

   g. $\sin A = 0.2300$ h. $\cos A = 0.9960$ i. $\tan A = 1.2000$

2. Find the angle between the diagonal and the base of a rectangle where the base is 8 cm and the height is 5 cm. (give your answer to the nearest degree)

Summary

1. For a right-angled triangle with hypotenuse h and other sides b and p, the Pythagoras theorem can be written as equation $b^2 + p^2 = h^2$.
2. Let ΔABC be a right-angled triangle with hypotenuse(H) and opposite side(O) and adjacent side (A), as shown in figure 5.11. Then, the trigonometric ratios are defined as

$$\sin A = \frac{\text{opposite}}{\text{hypotenuse}} = \frac{BC}{AC}$$
$$\cos A = \frac{\text{adjacent}}{\text{hypotenuse}} = \frac{AB}{AC}$$
$$\tan A = \frac{\text{opposite}}{\text{adjacent}} = \frac{BC}{AB}$$

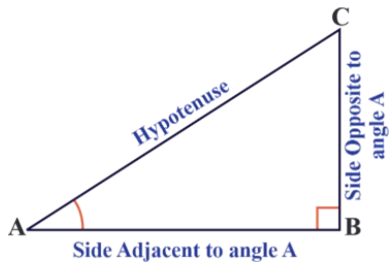


Figure 5.11

- For A and C as acute angle of the same right-angled triangle as shown in figure 5.11, $\sin A = \cos C$ and $\sin C = \cos A$.

3. The trigonometric values of basic angles $(0°, 30°, 45°, 60°$ and $90°)$ are given as:

➢ $\sin 0° = 0, \sin 30° = \frac{1}{2}, \sin 45° = \frac{1}{\sqrt{2}}, \sin 60° = \frac{\sqrt{3}}{2}$ and $\sin 90° = 1$

➢ $\cos 0° = 1, \cos 30° = \frac{\sqrt{3}}{2}, \sin 45° = \frac{1}{\sqrt{2}}, \cos 60° = \frac{1}{2}$ and $\cos 90° = 0$

➢ $\tan 0° = 0, \tan 30° = \frac{1}{\sqrt{3}}, \tan\ 45° = 1, \tan 60° = \sqrt{3}$ and $\tan 90°$ is undefined.

4. The following steps could be taken in order to read values of trigonometric ratio of sine, cosine and tangent from a trigonometric table.

- Identify the trigonometric ratio from the first row if the degree you need is between 0° and 45° and the last row if the degree is between 45° and 90°.
- Using the column of the trigonometric ratio you need to find and move down or up till you get the required degree.
- The intersection of the row and column of the above two steps is the trigonometric value of the required degree.

Review Exercise

1. Which one of the following is not correct about a right-angled triangle?
   - A. The angle opposite to the hypotenuse is a right angle
   - B. .If the base, height and hypotenuse of a right-angled triangle has lengths b, p and h units,respectively, then $b^2 + p^2 = h^2$.
   - C. For an isosceles right-angled triangle, two sides of a triangle are equal in length.
   - D. If θ is one of the angles of a right-angled triangle, then $\cos\theta$ can be greater than 1.
2. In ΔABC, right-angled at B, $AB = 24$ cm , $BC = 7$cm. Determine
   - a. $\sin A$, $\cos A$
   - b. $\sin C$, $\cos C$
3. State whether the following are true or false. Justify your answer.
   - a. The value of $\tan A$ is always less than 1.
   - b. $\sin\theta = \frac{4}{3}$, for some acute angle θ.
   - c. If $\sin\theta = \frac{1}{3}$, then $\cos\theta = \frac{2\sqrt{2}}{3}$.
   - d. When $0° \leq \theta \leq 90°$ is an angle of a right-angled triangle, both $\sin\theta$ and $\cos\theta$ are between 0 and 1.
4. Given $0° < \theta < 90°$. If $\tan\theta = 1$, then which one of the following is not true?
   - A. $\cos\theta = \frac{\sqrt{2}}{2}$
   - B. $\cos\theta = \sin\theta$
   - C. $\theta = 45°$
   - D. $\tan\theta = \sin\theta$
5. If $\sin\theta = \frac{2}{3}$ for an acute angle θ, then which one of the following is correct?
   - A. $\cos\theta = \frac{\sqrt{5}}{2}$
   - B. $\tan\theta = \frac{2}{\sqrt{5}}$

C. $\cos(90° - \theta) = \frac{3}{2}$ **D.** $\sin\theta > \cos\theta$

6. Find the value of each of the following using trigonometric table.

a. $\sin 18°$

b. $\cos 57°$

c. $\tan 71°$

7. Find the measure of an acute angle A, correct to the nearest degree if $\tan A = 12.10$.

8. The angle formed by the top of the building at a distance of 50 m from its foot on horizontal plane is found to be 60 degree. Find the height of the building. (See figure 5.12)

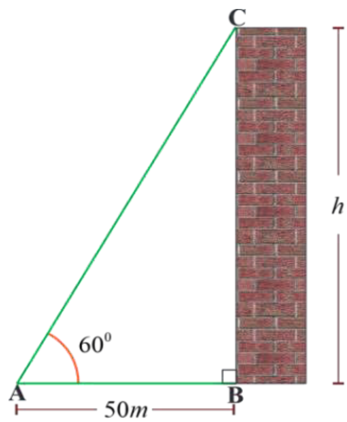


Figure 5.12

9. A building with its height of 60 m above sea level is located at Bahir Dar city near to Lake Tana. The angle between the line segment from the sailing boat to the top of the building and surface of the water is 25° as shown in figure 5.13.

a. How far is the boat from the base of the building to the nearest meter?

b. What is the distance between the man at the top of the building and the boat? (Use fig 5.13)

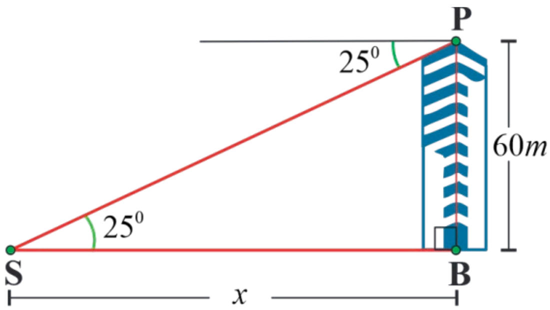


Figure 5.13

10. Two women A and B lie on the leveled ground at opposite sides of 150 m tall tower. If A observes the top of the tower at an angle of 60° and B observes the same point at an angle of 30°, then how far the two women can be away from each other?

A. $100\sqrt{3}$ m **B.** $600\sqrt{3}$ m

C. $200\sqrt{3}$ m **D.** $450\sqrt{3}$ m

11. In figure 5.14, if the height of the man is 1.74 m, $\theta = 63°$ and he is located at 100 m away from the building, find the height of the building.

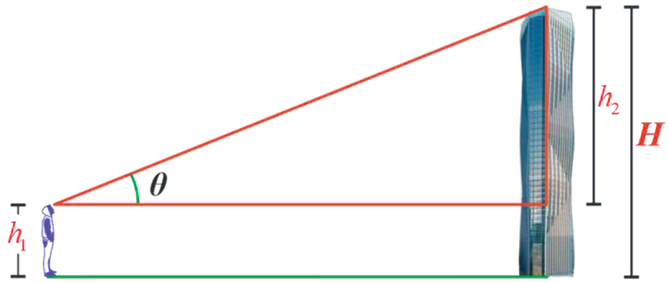


Figure 5.14

UNIT 6

REGULAR POLYGONS

Unit Outcomes

By the end of this unit, you will be able to:

- **Identify regular polygons.**
- **Solve problems on area and perimeter of regular polygons.**
- **Find the measure of each interior or exterior angle of a regular polygon.**
- **Explain properties of regular polygons.**

Unit Contents

KEY Words

- concave polygon
- area of regular polygon
- convex polygon
- perimeter of regular polygon
- in circle
- interior angles
- line of symmetry
- circumcircle
- exterior angles
- regular polygon
- apothem
- central angle

INTRODUCTION

In the previous grades, you studied different plane figures like triangles, quadrilaterals (rectangles, squares, and rhombus) and circles. You discussed how to get area and perimeter of such plane figures. In this unit, you will revise some of the basic concepts on polygons and study about regular polygon and their related properties.

6.1 Sum of Interior Angles of a Convex Polygon

Concave and convex polygons

Activity 6.1

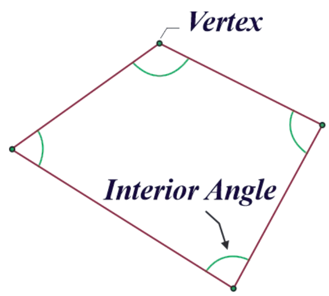
Answer each of the following.

1. How many sides and vertices (corners) does a triangle have?
2. How many sides and vertices (corners) does a quadrilateral have?
3. Can you draw a plane figure that has 5 sides and corners?

In the above activity, you discussed some plane figures. Now let us define a polygon.

Definition 6.1

A polygon is a simple closed plane figure formed by three or more line segments joined end to end, no two of which in succession are collinear. The line segments forming the polygon are called ***sides***. The common end point of any two sides is called ***vertex*** of the polygon. The angles formed inside the polygon are called ***interior angles***.



For instance:- Triangle (three sides) is the simplest polygon. Other polygons are quadrilaterals (four sides), and pentagons (five sides), and so on.

Common naming of polygon

In general, if a polygon has n sides, it is named as "n-gon", for instance a polygon with 16 sides is called 16-gon. The following table and figure provide the names of some polygons.

Table 6.1 Name of Polygons and Their Number of Sides and Vertices

Name of polygon	Number of sides	Number of vertices
Triangle	3	3
Quadrilateral	4	4
Pentagon	5	5
Hexagon	6	6
Heptagon	7	7
Octagon	8	8
Nonagon	9	9
Decagon	10	10

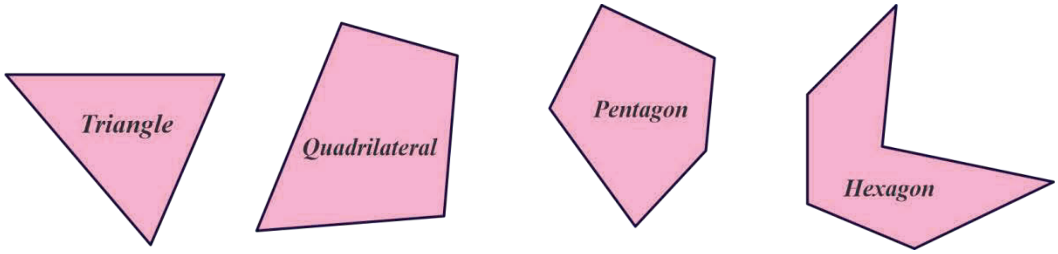


Figure 6.1 Polygons

Example

How many sides and vertices does a hexagon have?

Solution:

A hexagon is a polygon with 6 sides and 6 vertices.

Exercise 6.1

1. How many sides and vertices does a nonagon have?
2. What is the name of a polygon with ten sides?
3. Draw an octagon using a ruler.
4. When a triangle is cut off from a quadrilateral by a straight line which passes through vertices, write down the names of the possible polygons to be formed.

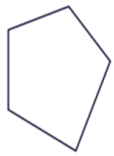
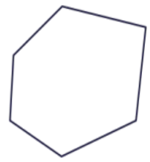
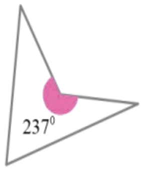
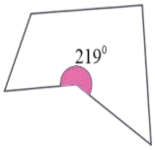
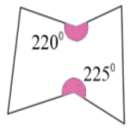
Definition 6.2

i. A polygon is said to be a convex polygon when the measure of each interior angle is less than 180 degrees. The vertices of a convex polygon are always outwards.

ii. A polygon is said to be a concave polygon when there is at least one interior angle whose measure is more than 180 degrees. The vertices of a concave polygon present inwards and also outwards.

Definition 6.2 could be easily observed by the following table 6.2.

Table 6.2

	Quadrilateral	Pentagon	Hexagon
Convex			
Concave			

The following could be another description for the above definitions.

Note

If none of the side extensions intersects the polygon, then the polygon is convex; otherwise it is concave.

Example

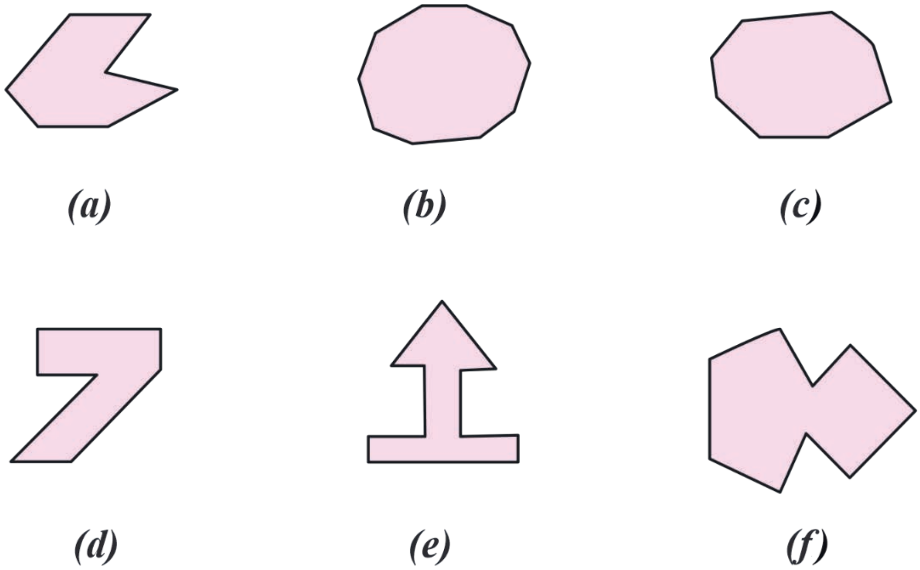
Is the quadrilateral in figure 6.1 convex or concave?

Solution:

The quadrilateral in figure 6.1 is convex since the measure of each interior angle is less than 180°. You can check this by measuring each angle using protractor.

Exercise 6.2

1. Is the hexagon in figure 6.1 convex or concave?
2. Name the following polygons and which of them are convex?



Sum of interior angles of a convex polygon

Activity 6.2

1. What is the sum of interior angles of a triangle?
2. What is the sum of interior angles of a quadrilateral?
3. Does the sum of interior angles of a polygon depend on the number of sides of the polygon?
4. How can you find the sum of interior angles of n-gon?

From your discussion in the activity and your previous grade geometry lessons, you learned that the sum of the measure of interior angles of any triangle is 180°.

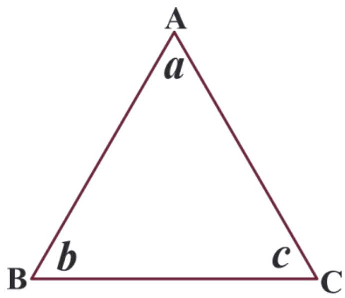


Figure 6.2

For any triangle ABC, note that

$$a + b + c = 180°\ .$$

Example 1

Determine the measure of interior angles of a quadrilateral.

Solution:

Consider a quadrilateral $ABCD$ as shown below in figure 6.3. Draw a line segment from one vertex, say A, to its opposite vertex C as shown below. Here, the line segment AC is one of the diagonals of the given quadrilateral.

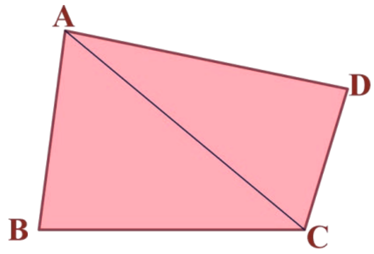


Figure 6.3

As you see from the figure 6.3 above, the quadrilateral is divided into two triangles so that the sum of interior angles of a quadrilateral is two times the sum of interior angles of a triangle. Hence, the sum of the measure of interior angles of a quadrilateral is 360°. Note that a quadrilateral can be divided into two triangles. This is $\times 180° = 2 \times 180°$, where $n = 2$, the number of triangles forming the quadrilateral.

Example 2

a. In how many triangles can we divide the pentagon?

b. What is the sum of the measures of interior angles of a pentagon?

Solution:

a. Consider a pentagon (a five-sided polygon). Draw a line segment (diagonal) from A to another vertex in the polygon (You do not have to draw lines to the adjacent vertices since they are already connected by a side) as shown in figure 6.4.

You can easily observe that the pentagon is divided into three triangles. So that the sum of interior angles of a pentagon is three times the sum of interior angles of a triangle.

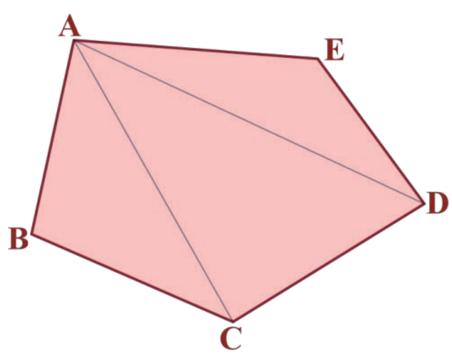


Figure 6.4

Note also that a pentagon can be divided into three triangles.

b. Sum of the measure of interior angles of a pentagon will be

$n \times 180° = 3 \times 180° = 540°$, where $n = 3$ is the number of triangles forming the pentagon.

Exercise 6.3

1. How many triangles can be made for a heptagon?
2. What is the sum of the measure of interior angles of a heptagon?

Deriving sum of the interior angles of a polygon

Can you derive a formula to determine the sum of the interior angles of a polygon?

Can you fill in the blank space of the following table?

Number of sides of a polygon	Name of the Polygon	Number of triangles	Sum of measure of interior angles
3	Triangle	1	$1 \times 180°$
4	Quadrilateral	2	$2 \times 180°$
5	Pentagon	3	$3 \times 180°$
6	Hexagon		
	.		
	.		
	.		
n	n-gon	$(n-2)$	$\_ \times 180°$

This indicates as the number of sides of a polygon increases, the number of triangles also increases. Furthermore, the sum of measure of interior angles of the polygon increases.

To summarize the above procedure, if n is the number of sides of a polygon, then $n - 2$ non-overlapping triangles are formed. Hence, we conclude the following statement:

Sum of the measure of interior angles of n-sided polygon is equal to $(n-2)\times 180°$.

Example 1

Find the sum of the measure of interior angles of hexagon.

Solution:

Hexagon is a 6-sided polygon. We use $n = 6$ in the above formula. Hence, the sum of measure of interior angles of a hexagon is equal to

$$(n-2)\times 180° = (6-2)\times 180° = 4\times 180° = 720°.$$

Example 2

The sum of interior angle of a polygon is 1,080°. How many sides does the polygon have?

Solution:

We need the value of n which satisfies $(n-2)\times 180° = 1{,}080°$.

$n-2 = \frac{1{,}080°}{180°} = 6$. So, $n = 8$. The polygon is octagon.

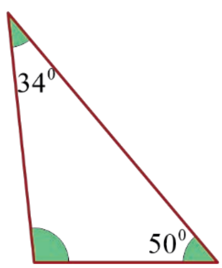
Exercise 6.4

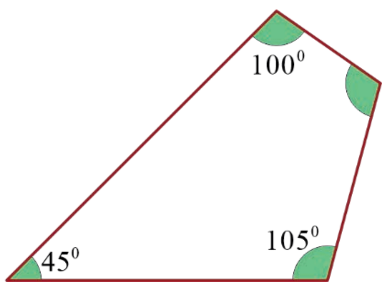
1. Find the sum of the measure of interior angles of:

   a. Nonagon b. a 12-sided polygon

2. The sum of interior angles of a polygon is 1,440° . How many sides does the polygon have?

3. Find the incomplete measure of interior angle of the given polygons.





4. An architect design to construct recreation area of a school with a heptagonal shape with each interior angle are put in increasing order, each differs from the next by 25°. Find the measure of the smallest interior angle of the given heptagon to the nearest tenth of degree.

6.2 Sum of Exterior Angles of a Convex Polygon

In the previous section, you have studied about convex polygons and how to determine the sum of the measure of interior angles of a polygon. In this section, you will learn about exterior angles of convex polygons.

Activity 6.3

Given triangle ABC as shown in the figure 6.5. Construct exterior angles of the triangle at each vertex.

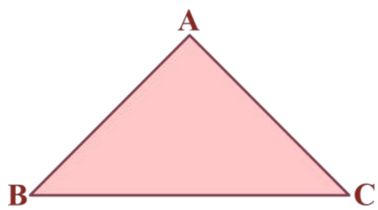
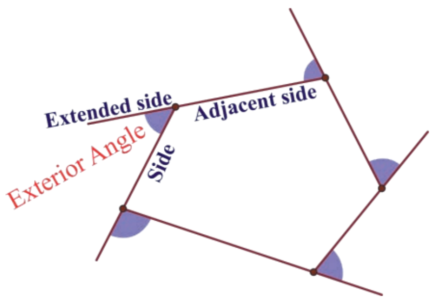


Figure 6.5

Definition 6.3

An exterior angle of a convex polygon is an angle outside the polygon formed by one of its sides and the extension of an adjacent side.



Example

Which ones from a to e are the exterior angles of the hexagon as shown in figure 6.6?

Solution:

By the definition, an exterior angle is an angle formed outside of the polygon which is formed by extending one of the sides with the next side as defined in definition 6.3. Hence, a and d are exterior angles, whereas e is not an exterior angle since it does not satisfy the definition.

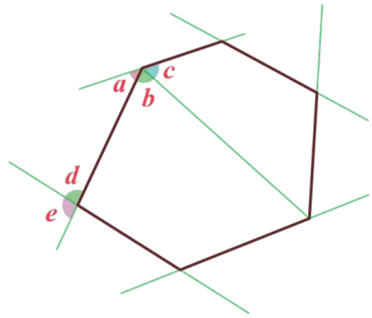


Figure 6.6

Exercise 6.5

1. Using a ruler, construct a heptagon and show its exterior angles.
2. Write **true** if the statement is correct and **false** otherwise.
   a. For a triangle, there are 6 exterior angles.
   b. For convex polygon, each exterior angle is less than $180°$.
   c. Each exterior angle of a rectangle is $90°$.
   d. If the measure of each interior angle of a regular polygon is $x°$, then the measure of each exterior angle of the given regular polygon is $(360 - x)°$.

Sum of the exterior angles of a polygon

What can you say about the sum of exterior angles of a polygon?

Does the sum depend on the number of sides of the polygon?

Let us see how to find sum of exterior angles of a polygon.

Example 1

Determine the sum of the exterior angles of a triangle.

Solution:

Consider a triangle ABC as figure 6.7. You will observe that angles α, β and γ are exterior angles, and angles a, b and c are

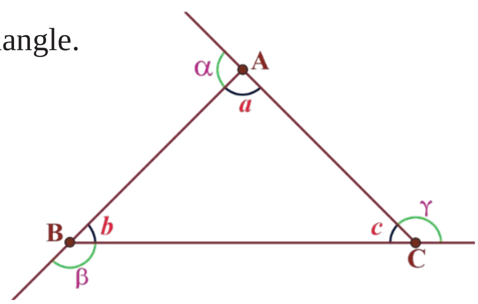


Figure 6.7

interior angles of ΔABC.

The sum of the interior angles of a triangle is 180°. $a+b+c=180°$. Also, we note that $(a+b+c)+(\alpha+\beta+\gamma)=3\times180°$ (measure of three straight lines). This leads to $\alpha+\beta+\gamma=360°$.

Hence, the sum of the measure of exterior angles of a triangle is 360°.

Example 2

Find the sum of the measure of exterior angles of a quadrilateral.

Solution:

Consider a quadrilateral $ABCD$ as shown in the figure 6.8 below with exterior angles α, β, γ and δ . Its interior angles are a, b, c and d. The sum of interior angles of a quadrilateral is $(n-2)\times180°=2\times180°=360°$ where n is the number of sides of the polygon. For our case, we know that $n=4$.

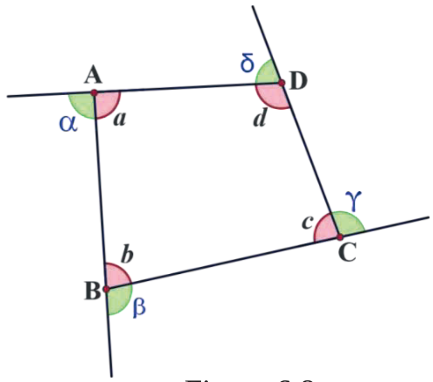


Figure 6.8

Hence, $a+b+c+d=360°$. From the figure we conclude that $(a+b+c+d)+(\alpha+\beta+\gamma+\delta)=4\times180°=720°$ and $\alpha+\beta+\gamma+\delta=360°$.

Hence, the measure of the exterior angles of a quadrilateral is 360°.

Exercise 6.6

1. Find the sum of the measure of exterior angles of a hexagon.
2. The exteriors angles of a pentagon are $(n+5)°$, $(2n+3)°$, $(3n+2)°$, $(4n+1)°$ and $(5n+4)°$ respectively. Find the measure of each angle.

Sum of the exterior angles of n-sided polygon

Similarly, if we take an n-sided polygon, the sum of the interior angles of a polygon is $(n-2)\times 180°$.

For such a polygon, we also have n straight lines. Each one is the sum of interior and exterior angle. Therefore, we have:

> The sum of interior and exterior angles of n-sided polygon is $n \times 180^0$.

Using this concept, we can derive the sum of exterior angles of a polygon as follows.

For an n-sided polygon,

Sum of interior angles of a polygon + sum of exterior angles of a polygon

$$= n \times 180° \qquad \text{(i)}$$

Using $(n-2)\times 180°$ in place of sum of interior angles of a polygon, equation (i) will be

$(n-2)\times 180°$ + sum of exterior angles of a polygon = $n \times 180°$. (ii)

Using distribution on equation (ii) we get

$n \times 180° - 360°$+ sum of exterior angles of a polygon = $n \times 180°$. (iii)

Collecting like terms and rearranging equation (iii) gives us

Sum of the measure of exterior angles of a polygon = 360°. (iv)

Hence, we can conclude that:

> For any n-sided polygon, the sum of the measure of exterior angles of the polygon is 360°.

Note

At each vertex of a polygon, the sum of interior and exterior angles is 180°.

Example

Consider a triangle whose two interior angles are 60° and 40°. Then, find

a. the remaining interior angle of the triangle.

b. the measure of exterior angle at each vertex.

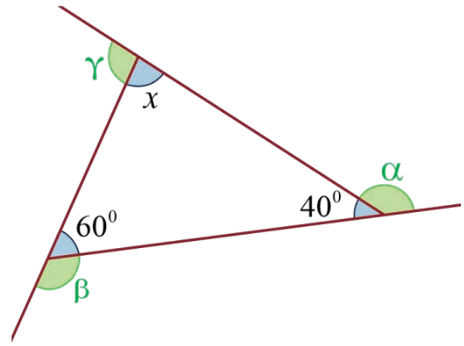


Figure 6.9

Solution:

a. Consider a triangle with two of its interior angles are 60° and 40° as shown in figure 6.9 . We know that the sum of the measure of interior angles of a triangle is 180°. Assume the degree measure of the remaining interior angle of the triangle be x. Since the sum of the measure of interior angles of a triangle is 180°, then $x + 60° + 40° = 180°$. $x = 180° - (60° + 40°) = 80°$.

b. We also know that a straight line measures 180°, then the exterior angle

$\alpha = 140°$, $\beta = 120°$ and $\gamma = 100°$.

Is the sum of these exterior angles in Example 1, 360°? Yes,

$100° + 120° + 140° = 360°$.

Exercise 6.7

1. Given a pentagon with four of its interior angles are 120° , 95°, 100° and 70°. Find the measure of exterior angle at each vertex.

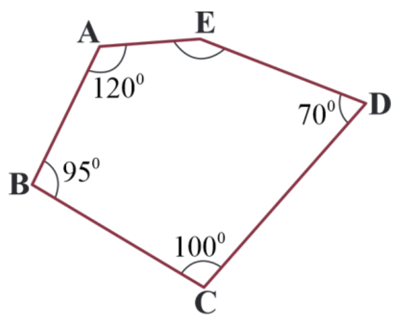


Figure 6.10

2. Find the measure of angle x and y from the following figure 6.11. The dashes on each line are to mean the line segments are equal in length.

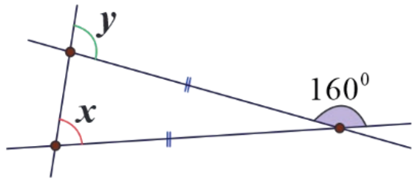


Figure 6.11

6.3 Measures of Each Interior Angle and Exterior Angle of a Regular Polygon

In the previous two subsections you learned how to determine the sum of the interior and exterior angles of a polygon. In this section you will see the definition of a regular polygon, and how to determine each interior and exterior angle of such polygon.

A summary of Greek mathematics was given by Euclid in about 300 BCE in his work the *Elements*. Euclid taught in Alexandria, Egypt, and wrote a number of mathematical treatises. His *Elements* consists of thirteen books and includes topics on number theory and irrationals as well as geometry.

Euclid gave constructions and proofs for an equilateral triangle (Proposition 1 in Book I), the square and pentagon (Book IV). He also shows that using the triangle and pentagon a 15-sided polygon can be constructed; furthermore, by bisecting angles further polygons can be constructed from all these, e.g. 6-,12-,24-,48-sided n-gons .

Euclid (325 BCE-265 BCE)

Activity 6.4

What common property do you observe for the following in figures 6.12?

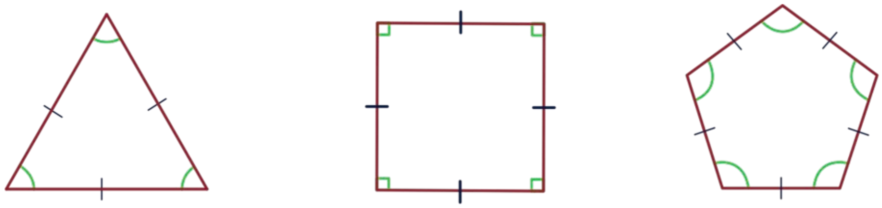


Figure 6.12

For a polygon of the above types, we have the following definition:

Definition 6.4

A polygon is said to be regular if it is equiangular (having equal angles) and equilateral (having equal sides).

For example: -Equilateral triangles and squares are regular polygons. Students, do you know honeycomb? We also have such polygons in our day-to-day life. For instance, figure 6.13 shows a honeycomb which is formed by honeybees as regular hexagons.

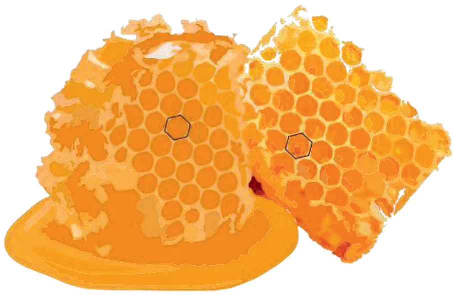


Figure 6.13

What is the measure of each interior and exterior angle of n-sided regular polygon?

For n-sided polygon, recall that

i. The sum of the measure of interior angles is $(n - 2) \times 180°$.

ii. The sum of the measure of exterior angles is $360°$.

Using this and the definition of regular polygon, the measure of each interior angle of n-sided regular polygon is determined by

$\frac{(n-2)\times 180°}{n}$ (Since a regular polygon is equiangular) and the measure of each exterior angle is $\frac{360°}{n}$ (since there are n exterior angles with same measure).

Example

Find the measure of each interior and each exterior angle of a regular pentagon.

Solution:

Given a regular pentagon which is five-sided ($n = 5$). So each interior angle degree measure is

$$\frac{(n-2)\times 180°}{n} = \frac{(5-2)\times 180°}{5} = \frac{3\times 180°}{5} = 108°$$

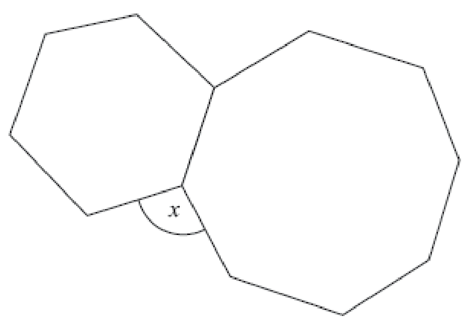
The measure of each exterior angle is $\frac{360°}{5} = 72°$.

Exercise 6.8

1. Find the measure of each interior and exterior angle of

   i) a regular hexagon   ii) a regular octagon

2. What is the sum of each interior and its corresponding exterior angle of the above two regular polygons?
3. A Mathematics teacher gave a group work for students to construct a convex regular polygon with one of the interior angle measures 135°. What is the name of this polygon?
4. For n-sided regular polygon, the measure of each interior angle is 5 times the measure of each exterior angle. What is the number of sides of this polygon?
5. The diagram shows a regular hexagon and a regular octagon. Calculate the size of the angle marked x.



6.4 Properties of Regular Polygons: Pentagon, Hexagon, Octagon and Decagon

In section 6.3, you studied about regular polygon, interior and exterior angles of such polygons. In this section, you will discuss the basic properties of regular polygons.

Line of symmetry for regular polygons

You have learned about symmetry of simple polygons in lower grades. You will apply the concept of symmetry to do the following activity.

Activity 6.5

On a sheet of paper draw an ***equilateral triangle*** and ***isosceles triangle***. Then cutout the above polygonal region and practice the following (you may need to use ruler, protractor and pencil.)

1. For each of the above polygonal region, how many lines of symmetry can you get? (Fold it in different ways till you obtain the folding line which serves as a mirror or line of symmetry)
2. Is there a relation between the number of lines of symmetry and the sides of a polygon?

Now, let us discuss symmetry of regular polygons.

- If two parts of a figure are identical after folding or flipping, then it is said to be ***symmetric***. To be symmetrical, one half must be the mirror image of the other. If the figure is not symmetrical then it is said to be ***asymmetrical***.
- ***Line of symmetry*** is a line through which the two halves of the figure match exactly. It is also called ***mirror line***. A figure can have more than one line of symmetry. An asymmetrical figure has no line of symmetry.

Note

All regular polygons are symmetrical shapes so that we have lines of symmetry for each.

The following figure shows the first four regular polygons with lines of symmetry.

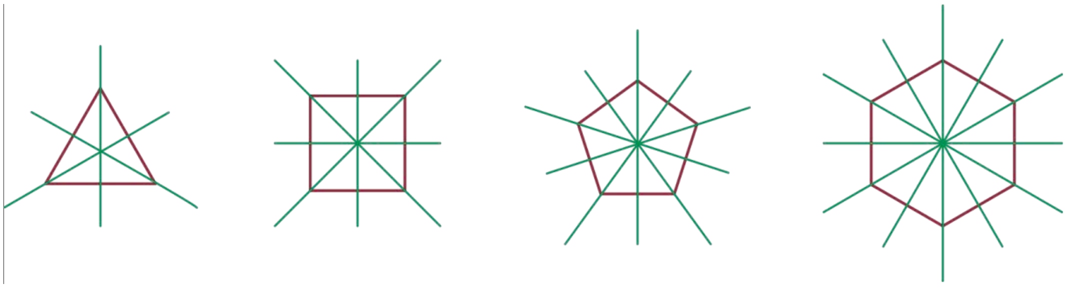


Figure 6.14: Some regular polygons with lines of symmetry

From figure 6.14 above, you can observe that

- An equilateral triangle has 3 lines of symmetry.
- A square has 4 lines of symmetry (two horizontal and vertical lines of symmetry, as well as two diagonal lines of symmetry).
- A regular pentagon has 5 lines of symmetry.
- A regular hexagon has 6 lines of symmetry.
- The lines of symmetry meet at a point inside the polygon called center of the polygon.
- If n is the number of sides of a regular polygon:
  - If n is odd, the line of symmetry connect vertex to side.
  - If n is even, the line of symmetry connect vertex to vertex and side to side.

From the above discussion, we can conclude the following: -

The n -sided regular polygon has n lines of symmetry.

Example

How many lines of symmetry does a regular heptagon have?

Solution:

A regular heptagon has 7 sides so that it has 7 lines of symmetry. It can be observed from the following figure 6.15.

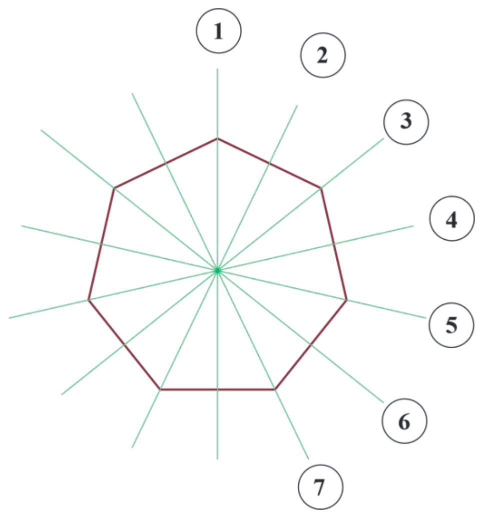


Figure 6.15

Exercise 6.9

1. How many lines of symmetry does a regular octagon have?
2. For each of the following statements write 'true' if the statement is correct and 'false' otherwise. If your answer is false give justification why it is false.
   a) If a polygon is not regular, then it is asymmetric.
   b) A right-angled triangle has at most one line of symmetry.
   c) A rhombus has 4 lines of symmetry.
   d) For n-sided regular polygon, the lines of symmetry divide the polygon into $2n$ triangles.

Inscribed and circumscribed polygon

Activity 6.6

1. Construct the following plane figures.
   a. A right-angled triangle with sides 6 cm and 8 cm that form the right angle.
   b. A square of side 10 cm.
   c. An equilateral triangle of side length 7 cm.
2. Using each of the above polygons,
   a. Try to draw a circle which passes through all vertices of the polygon.
   b. Try to draw a circle in the polygon where it touches all its sides.
3. Is it possible to draw such circles for the above polygons? (Identify the type of polygon for which such types of circles can be made).

In the above activity you observe facts about circles which could be made in and out of a polygon. Based on the discussion you had, we can study facts based on regular polygons. Let us consider n-sided regular polygon where O be the center of a circle and a polygon. In the figure below $\overline{AB}$, $\overline{BC}$, $\overline{CD}$,…are sides of n-sided regular polygon, E is the mid-point of $\overline{BC}$ which is one of the n sides of the polygon. Then we have:

- The n-sided polygon is inscribed in the bigger circle whose radius is $\overline{OB}$. Such a circle is called **a circumcircle**, and it connects all vertices (corner points) of the polygon.
- The line segment from the center perpendicular to the sides of a regular

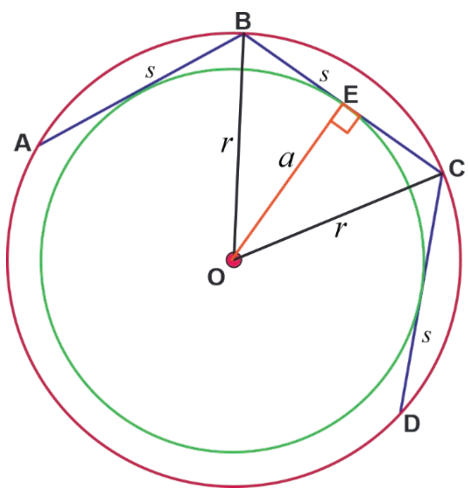


Figure 6.16

polygon is called an **apothem** of a polygon (for instance $a = \overline{OE}$ is an apothem which is perpendicular bisector of $\overline{BC}$).

- The inside circle (the smaller circle) is called an **in circle** and it just touches each side of the polygon at its midpoint. The radius of the incircle is the **apothem** of the polygon. (Not all polygons have these properties, but triangles and regular polygons do).
- A circle is inscribed in and circumscribed about any regular polygon.

Measure of central angles

The central angle for a regular polygon is an angle formed at the center of a circle by two consecutive radii from the vertex of a polygon (see figure 6.16, $\angle BOC$ is a central angle). For n-sided regular polygon, we can obtain n isosceles triangles (see also figure 6.16, ΔBOC is one of such isosceles triangles). Recall that the sum of measures of angles at a point is $360°$. Therefore, the sum of the central angles is $360°$. Hence, the measure of each central angle of n-sided regular polygon is $\frac{360°}{n}$ (each interior angle of a regular polygon has the same measure).

Example

Find the measure of central angle of a regular polygon inscribed in a circle which has the given number of sides:

a. 5 **b.** 8

Solution:

a. The measure of the central angle depends on the number of sides of a regular polygon. For $= 5$, the measure of each central angle is $\frac{360°}{5} = 72°$.

b. For $n = 8$, the measure of each central angle is $\frac{360°}{8} = 45°$.

Exercise 6.10

1 Determine the measure of each central angle of a regular polygon which is

a. 10 sided b. 15 sided

2. Find the number of sides of a regular polygon whose measure of central angle is $12°$.

3. Which of the following measure of central angles yields a regular polygon?

a. $6°$ b. $14°$ c. $80°$ d. 40°

Perimeter, area and apothem of regular polygons inscribed in a circle

There is a close relationship between the perimeter, area, side length and apothem of an n-sided regular polygon which is inscribed in a circle. We discuss a particular case as an illustrative example and we develop the formula through the process for the general n-sided regular polygon.

Example

In figure 6.17, the regular hexagon is inscribed in a circle with center O and radius r. Express the side length s, perimeter P, apothem a, and area A of the regular hexagon using r.

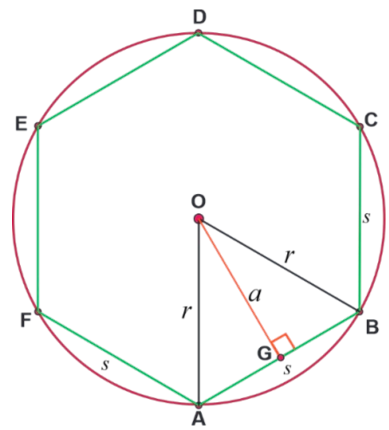


Figure 6.17

Solution:

Given that ABCDEF is a regular hexagon. It has 6 sides. Triangle AOB is one of equilateral triangle as shown in figure 6.17.

(since $\angle AOB = 360° \div 6 = 60°$,

$\angle OAB = \angle OBA = (180° - 60°) \div 2 = 60°$)

There are 6 congruent equilateral triangles inside this regular hexagon. So that $\underline{s = r}$, $\underline{P = 6s = 6r}$.

Draw a perpendicular bisector from O to $\overline{AB}$ = s. It meets $\overline{AB}$ at G. $\overline{OG}$ is the apothem a of the hexagon.

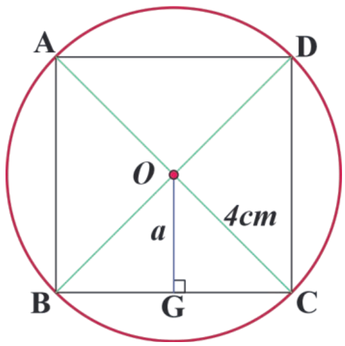
Here $\Delta AOG \equiv \Delta BOG$ (SSS). So that, m (∠AOG) = m (∠BOG) = 60° ÷ 2 = 30°.

Then, $a = r\cos 30° = \frac{\sqrt{3}}{2}r$. The area of $\Delta AOB = \frac{1}{2}\ \overline{AB} \times \overline{OG} = \frac{1}{2}ar = \frac{\sqrt{3}}{4}r^2$.

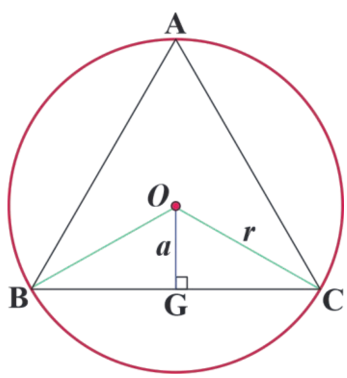
The area of the pentagon ABCDEF $= A = 6 \times \Delta AOB = 6 \times \frac{\sqrt{3}}{4}r^2 = \frac{3\sqrt{3}}{2}r^2$.

Exercise 6.11

1. The figure below shows that a square is inscribed in a circle with the radius of 4 cm. Find the length of the side, the apothem and the perimeter, and the area of the square.



2. The figure below shows that an equilateral triangle is inscribed in a circle with the radius of r. Show the length of the side, the apothem, the perimeter, and the area of the triangle in terms of the radius r.



Example 1

Suppose a regular polygon (n-gon) is inscribed in a circle with center O and radius r. Find the side length s, perimeter P, apothem a, and area A of the regular n-gon. (In this case, n-gon is not necessarily a hexagon, but Fig 6.17 may help our understanding).

A regular n-gon is given in the figure. It has n sides. Triangle AOB is one of such isosceles triangles as shown in figure 6.16. Moreover, there are n congruent isosceles triangles inside this regular n-gon. Triangle AOB is one of such isosceles triangles as shown in figure 6.17.

Draw a perpendicular bisector from O to $\overline{AB}$ at G.

$\overline{OG}$ is the apothem a of the n-gon. Here, $\Delta AOG \equiv \Delta BOG$ (SSS).

And $m(\angle AOB) = 360° \div n = \frac{360°}{n}$

So that, $m(\angle AOG) = m(\angle AOG) = \frac{360°}{n} \div 2 = \frac{180°}{n}$

Then, $\frac{s}{2} = r\sin(\frac{180°}{n})$ which implies $\underline{s = 2r\sin(\frac{180°}{n})}$.

$\underline{P = n \times s = 2nr\sin(\frac{180°}{n})}$ and $\underline{a = r\cos\left(\frac{180°}{n}\right)}$.

The area of $\Delta AOG = \frac{1}{2}\ \overline{AB} \times \overline{OG} = \frac{1}{2}\ sa$. So, the area of the n-gon is

$A = n \times$ area of isosceles triangle. This is $A = \frac{1}{2}nas$, but $P = ns$, so that $\underline{A = \frac{1}{2}aP}$.

The above example 1 leads us to state the following theorem.

Theorem 6.1

For n-sided regular polygon inscribed in a circle of radius r, length of side s, apothem a, perimeter P and area A are determined by

$$s = 2r\sin\left(\frac{180°}{n}\right) \qquad a = r\cos\left(\frac{180°}{n}\right)$$

$$P = 2nr\sin\left(\frac{180°}{n}\right) \qquad A = \frac{1}{2}aP$$

Example 2

a. Find the length of side of a square inscribed in a circle of radius 5 cm.

b. Find the apothem of a regular pentagon inscribed in a circle of radius 6 cm.

c. Find the area of a regular nonagon inscribed in a circle whose radius is 10 cm.

(Use a calculator or a trigonometric table whenever necessary)

Solution:

a. By the formula in the above theorem 6.1, $s = 2r\sin\left(\frac{180°}{n}\right)$. Using $r = 5$ cm and $n = 4$, the length of the side is

$$s = 2r\sin\left(\frac{180°}{n}\right) = 2(5\text{ cm})\sin\left(\frac{180°}{4}\right)$$
$$= 10\text{ cm} \times \sin(45°)$$
$$= 10\text{ cm} \times \frac{1}{\sqrt{2}} = 5\sqrt{2}\text{ cm} \quad \text{(rationalizing the denominator)}$$

b. Given $r = 6$ cm and $n = 5$, we can determine the apothem (a) by using the formula $a = r\cos\left(\frac{180°}{n}\right)$, $a = 6\text{ cm} \times \cos\left(\frac{180°}{5}\right) = 6\text{cm} \times \cos(36°)$. Using scientific calculator or trigonometric table $\cos(36°) \approx 0.8090$.

Hence, $a \approx 4.854$ cm

c. The area of a regular polygon is determined by

$$\boldsymbol{A} = \frac{1}{2}aP = \frac{1}{2}\left(r\cos\left(\frac{180°}{n}\right)\right) \times 2nr\sin\left(\frac{180°}{n}\right)$$

Using $n = 9$ and $r = 10$ cm,

$$\boldsymbol{A} = nr^2\cos\left(\frac{180°}{n}\right)\sin\left(\frac{180°}{n}\right) = 9(100\text{ cm}^2)\cos(20°)\sin(20°)$$

(Using trigonometric table/Scientific calculator)

$\cos(20°) \approx 0.9397$ and $\sin(20°) \approx 0.3420$, hence, $A = 289.240\text{ cm}^2$.

Exercise 6.12

1. Find the perimeter, area and apothem of a decagon which is inscribed in a circle of radius $r = 8$ cm.
2. Suppose central angle of a regular polygon is 60°. Then find the radius of a circle circumscribing the given polygon if its side is of length 9 cm .
3. One student from your class states that 'the radius of a regular polygon is never less than its apothem'. Do you agree? If so provide justification. If not, provide a counter example.

Summary

1. A polygon is a simple closed plane figure formed by three or more line segments joined end to end, no two of which in succession are collinear. The line segments forming the polygon are called ***sides***. The common end point of any two sides is called ***vertex*** of the polygon. The angles formed inside the polygon are called ***interior angles.***
2. **i)** A polygon is said to be a convex polygon when the measure of each interior angle is less than 180°.
   ii) A polygon is said to be a concave polygon when there is at least one interior angle whose measure is more than 180 degrees.
3. The sum of the measure of interior angles of n-sided polygon is determined by $(n-2) \times 180°$.
4. For any n-sided polygon, the sum of the measure of exterior angles of the polygon is 360°.
5. At each vertex of a polygon, the external and internal angle add up to 180° .
6. A regular polygon is a convex polygon with all sides equal and all interior angles equal.
7. **i)** The measure of each interior angle of a regular n-sided polygon is $\frac{(n-2)\times 180°}{n}$.
   ii) The measure of each exterior angle of a regular n- sided polygon is $\frac{360°}{n}$.
   iii) The measure of each central angle of a regular n-sided polygon is $\frac{360°}{n}$.
8. If two parts of a figure are identical after folding through the line of symmetry, then it is said to be symmetric. A symmetrical figure has at least one line of symmetry. An asymmetrical figure has no line of symmetry.
9. An n-sided regular polygon has n lines of symmetry.

10. An inscribed polygon is a polygon in which all vertices lie on a circle. The polygon is **inscribed** in the circle and the circle is **circumscribed** about the polygon.

11. A circumscribed polygon is a polygon in which each side touches a circle. The circle is **inscribed** in the polygon and the polygon is **circumscribed** about the circle.

12. A circle can always be inscribed in or circumscribed about any regular polygon.

13. An apothem of a regular polygon is a perpendicular segment from a midpoint of a side of a regular polygon to the center of the circle circumscribed about the polygon.

14. Formula for the length of side s, apothem a, perimeter P and area A of the regular polygon with n-sides and radius r are given by

$$s = 2r\sin\left(\frac{180°}{n}\right) \qquad a = r\cos\left(\frac{180°}{n}\right)$$

$$P = 2nr\sin\left(\frac{180°}{n}\right) \qquad A = \frac{1}{2}aP$$

Review Exercise

1. Write true if the statement is correct and false otherwise.

a. If a polygon is equiangular, then it is equilateral.

b. Equiangular hexagons are regular.

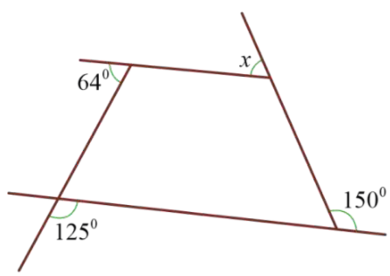
c. All symmetric plane figures are polygons.

d. If a polygon is inscribed in a circle, then the polygon is regular.

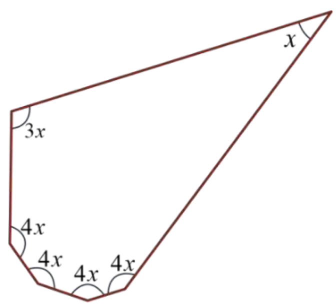
e. Fixing the radius of a polygon inscribed in a circle, increasing the number of sides of a regular polygon increases the length of the apothem.

f. If a regular polygon of n sides has every line of symmetry passing through a vertex, then n is even.

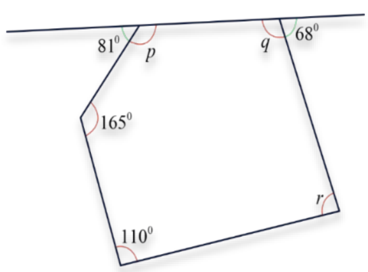
2. Find the sizes of the angles marked by letters.



(a)



(b)



(c)

3. Find the number of sides of a regular polygon if each exterior angle is equal to
   i) its adjacent interior angle.
   ii) twice its adjacent interior angle.
4. Is it possible to construct a polygon whose sum of interior angles is 26 right angles? If yes, find the number of sides of the polygon.
5. One of the interior angles of a polygon is $100°$ and each of the other angles is $110°$. Find the number of sides of the polygon.
6. The interior angles of a polygon are in the ratio 2:3:5:9:11. Find the measure of each angle. Is the polygon convex or concave? Why?
7. Each side of a regular hexagon is 10 units long. What is the area of this hexagon?
   A. 27 unit^2 B. 75 unit^2 C. $150\sqrt{3}$ unit^2 D. $54\sqrt{3}$ unit^2
8. The area of a regular hexagon is $96\sqrt{3}\ \text{cm}^2$. What is its perimeter?
   A. 48 cm B. 54 cm C. 42 cm D. 60 cm
9. Which one of the following statements is correct about a regular octagon?
   A. From one vertex we can draw 4 diagonals only.
   B. The sum of the measures of all its interior angles is1080° .
   C. The sum of the measures of all its central angles is 180°.
   D. The measure of each exterior angle is 135°.
10. Which one of the following is not correct about n-sided regular polygon?
   A. It has n lines of symmetry.
   B. The measure of each interior angle is $\frac{(n-2)\times 180^\circ}{n}$.
   C. The measure of each central angle and each exterior angle is $\frac{360^\circ}{n}$.
   D. The sum of the measure of exterior angles is $n \times 360^\circ$.
11. If the measure of each exterior angle of a regular polygon is 40°, then how many sides does this polygon have?

A. 7 B. 9 C. 8 D. 10

12. The perimeter of regular pentagon inscribed in a circle with radius 10 cm is equal to :

(use $\sin 36^\circ = 0.59$, $\cos 36^\circ = 0.81$, $\sin 72^\circ = 0.95$, $\cos 72^\circ = 0.31$)

A. 81 cm B. 50 cm C. 91 cm D. 59 cm

13. Is it possible to have a regular polygon with each of whose exterior angle is 50° ? Give reason to support your answer.

14. What is the perimeter of a square with a diagonal of $5\sqrt{2}$cm?

15. How many sides does a regular polygon have if the measure of an interior angle is 165°?

16. There are two regular polygons with number of sides equal to $(n - 1)$ and $(n + 2)$. Their exterior angles differ by 6°. Find the number of sides of the two polygons.

17. The figure (fig 6. 18) shows two regular pentagons and an equilateral triangle(ΔEDF). Determine the measure of $\angle AEI$.

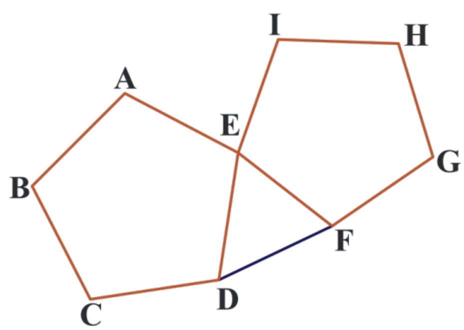


Figure 6.18

18. A regular polygon has a perimeter 143 unit and the sides are 11 unit long. How many sides does the polygon have?

19. Suppose an equilateral triangle is inscribed in a circle of radius 8 cm. Find the area of the region which is in a circle and outside the triangle.

(use $\pi \approx 3.14$, $\sqrt{3} \approx 1.73$)

20. Find the possible measure of minimum interior angles and maximum exterior angles in a regular polygon. Give a reason to support your answer.

21. The school pedagogical center is making a design of a regular octagonal stop sign (as shown in fig. 6.19) which will be placed at the main gate of the school to prevent car accidents. Each side of the sign is 40cm long. What is the area of the sign? (to the nearest integer)

Figure 6.19

22. A gardener has enough material for 660 meters of fence. She knows that a circle maximizes the area to perimeterration, but circular fence is difficult to construct. So, she wants to simulate a circle by constructing a fence in a shape of a regular polygon with as many sides as possible but with more than 5 and not more than 15 sides. If she wishes to make each side a length of an even meter, and she wishes to have no more than 2 meter left over, which regular polygon would you suggest she build, and how long would each side be?

A. Octagon; 83 m **B.** 13-gon; 51 m **C.** 15-gon; 44 m **D.** 11-gon; 60 m

UNIT 7

CONGRUENCY AND SIMILARITY

Unit Outcomes

By the end of this unit, you will be able to:

- Distinguish between congruent and similar plane figures.
- Apply postulates and theorems in order to prove congruency and similarity of triangles.
- Solve problems on perimeters and areas of similar triangles and similar polygons.
- State and use the criteria for similarity of triangles viz. AAA, SSS and SAS.
- Prove the Pythagoras Theorem.
- Apply these results in verifying experimentally (or proving logically) problems based on similar triangles.

Unit Contents

KEY Words

- Congruency
- SAS similarity
- Plane figures
- Proportionality
- AA similarity
- Ratio of area of similar plan figure
- Ratio of perimeter of similar plan figure
- Construction of similar plane figures
- Similarity
- SSS similarity
- triangles
- Application of similarity

Introduction

There are maps of different countries in the world which are smaller or larger in their size or same shape or not. If you ask yourself whether a country's different maps are like or unlike, equal in size or not, you may answer it by looking at the maps. In geometry, the concept of equality in size and shape is described as congruency of triangles and likeness is called "similarity of figures". In this unit, you will learn both congruency and similarity of triangles.

7.1 Revision on Congruency of Triangles

Activity 7.1

a. What can you say about the shape and size of congruency of two triangles?

b. Can you mention some conditions of congruency of triangles?

c. For ΔABC & ΔDEF, $\overline{AB} \equiv \overline{DE}$ and $\angle B$ has the same measure as $\angle E$. .
Which of the following statements, if true, is sufficient to show that the two triangles are congruent?

i. $\overline{AB} \equiv \overline{DE}$ iii. $\angle C \equiv \angle F$

ii. $\angle A \equiv \angle D$

What is “congruency”? It means that one shape can become another using turns, flips and/or sliders. The equal sides and angles may not be on the same position (if there is a turn or a flip), but they are there. Congruent means to be identical in size and shape.

Definition 7.1

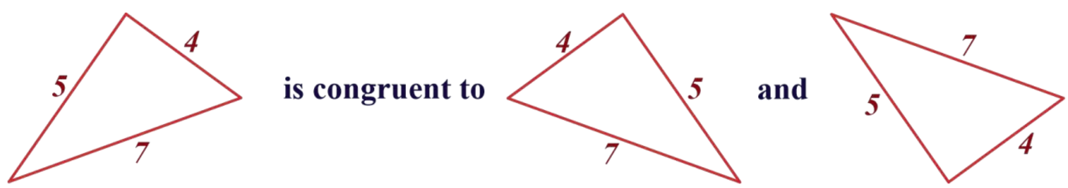
When two triangles have exactly the same three sides and the same three angles they are called congruent triangles.

Note

Same three sides means the corresponding sides have equal length and same three angles means the corresponding angles have equal measure.

Note that $\triangle ABC \equiv \triangle DEF$ means $\overline{AB} \equiv \overline{DE}$, $\overline{BC} \equiv \overline{EF}$, $\overline{AB} \equiv \overline{DE}$, $\overline{AC} \equiv \overline{DF}$ and $\angle A \equiv \angle D$, $\angle B \equiv \angle E$, $\angle C \equiv \angle F$

For example:



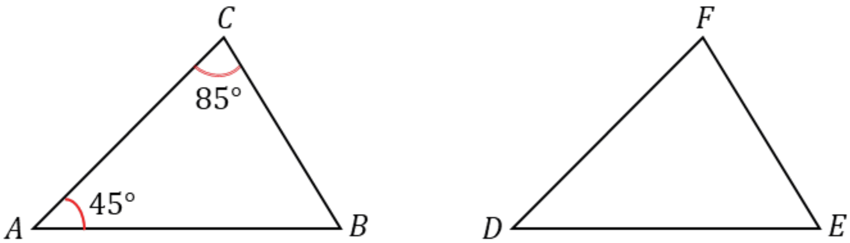
The above three triangles are congruent.

Exercise 7.1

a) If two triangles are congruent, then they have exactly same shape and size (True or False).

b) Any two equilateral triangles are congruent (True or False).

c) In the figure below, $\triangle ABC \equiv \triangle DEF$, $\angle A = 45°$, $\angle C = 85°$. Find the measures of $\angle D$ and $\angle E$.



1. SSS congruency

Side-Side-Side congruency: When two triangles have equal corresponding sides, then the triangles are congruent.

2. SAS congruency

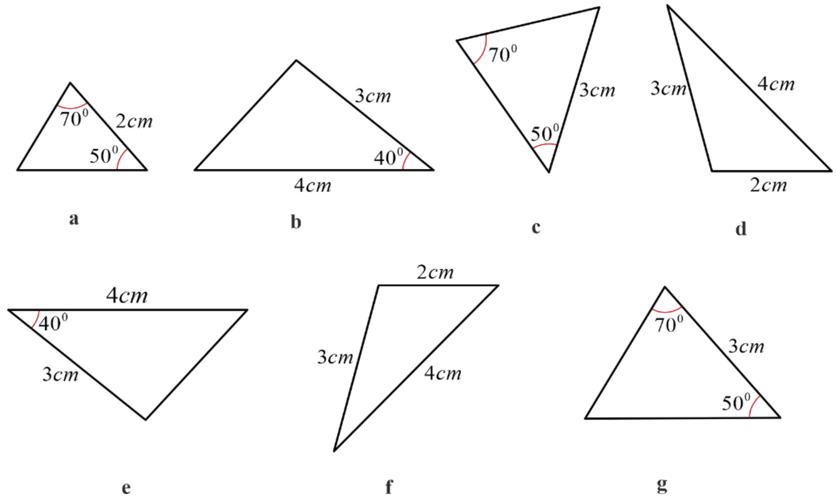
Side-Angle-Side congruency: When two triangles have two pair of congruent sides and one pair of congruent angles between the sides, then the triangles are congruent.

3. ASA congruency

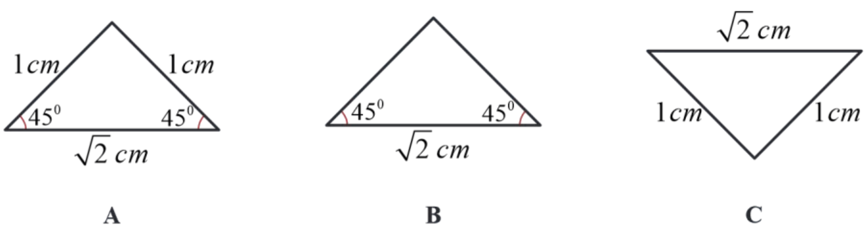
Angle-Side-Angle congruency: When two triangles have two pair of congruent angles and one pair of congruent sides between the angles, then the triangles are congruent.

Exercise 7.2

1. Identify which triangles are congruent? Why?

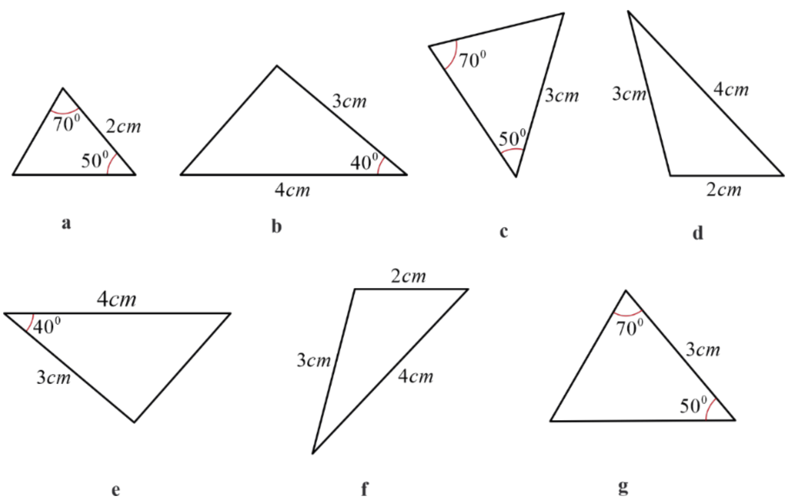


2. Which pair of triangles are congruent? Why?



Exercise 7.3

Which of the following triangles are congruent by ASA condition?

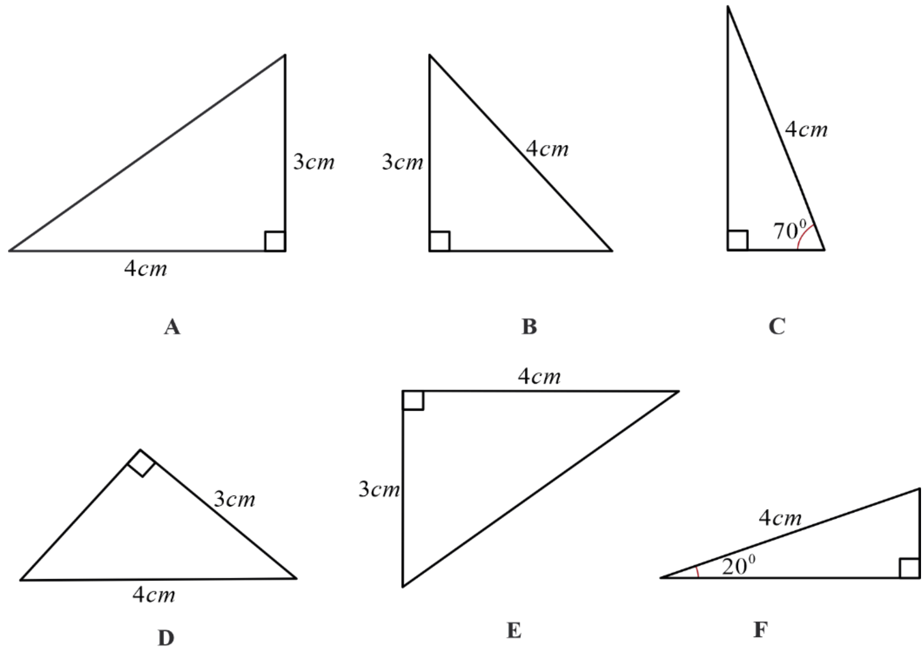


4. RHS Congruency

Right-angle hypotenuse side congruency: When two right triangles have equal hypotenuse and equal one side (leg), then the triangles are congruent.

Exercise 7.4

a) Identify the triangles into pairs of congruent figures.



b) Among the following statements, which the following conditions from (i) to (viii) will represent for ΔABC and ΔPQR to be congruent?

i. $AB \equiv PQ, BC \equiv QR, AC \equiv PR$

ii. $BC \equiv QR, \angle B \equiv \angle Q, \angle C \equiv \angle R$

iii. $AB \equiv PQ, BC \equiv QR, \angle C \equiv \angle P$

iv. $AB \equiv PQ, BC \equiv QR, \angle B \equiv \angle Q$

v. $AB \equiv PQ, \angle A \equiv \angle P, \angle C \equiv \angle R$

vi. $AB \equiv PQ, BC \equiv QR$

vii. $\angle B \equiv \angle Q, \angle C \equiv \angle R$

7.2 Definition of Similar Figures

Activity 7.2

1. If two plane figures have the same number of sides, what can you say about the plane figures?
2. If two plane figures have the same area, what can you say about similarity and congruency the plane figures?

Definition 7.2

Two plane figures are similar if their corresponding angles are congruent and the ratios of their corresponding sides are proportional. This common ratio is called the scale factor.

Two plane figures are similar means that

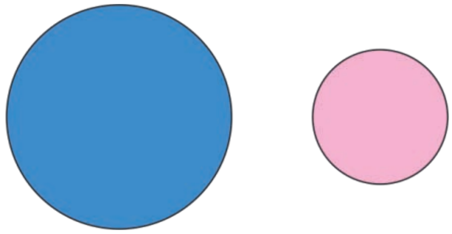
- All corresponding angles are congruent
- All corresponding sides are proportional

When we magnify or diminish similar figures, one exactly places on the other and vice versa.

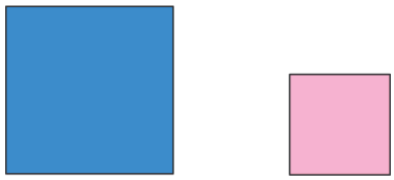
1. If the two triangles, the two rectangles and the two pentagons below satisfy the above properties then they are similar to each other.

   Note that if ΔABC and ΔDEF are similar, then we write mathematically as $\Delta ABC \sim \Delta DEF$.

2. Any two circles (of any radii) have the same shape and hence they are always similar.



Similarly, two squares (of any side lengths) are similar.



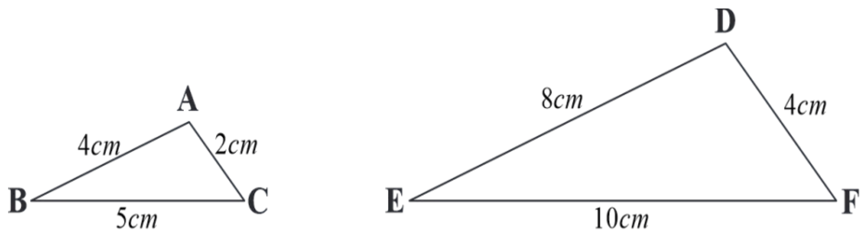
Exercise 7.5

Any two regular polygons are similar (True or False). Why?

Example 1

Similar triangles and quadrilaterals:

Given $\Delta ABC \sim \Delta\, DEF$, find the common ratio of their corresponding sides.



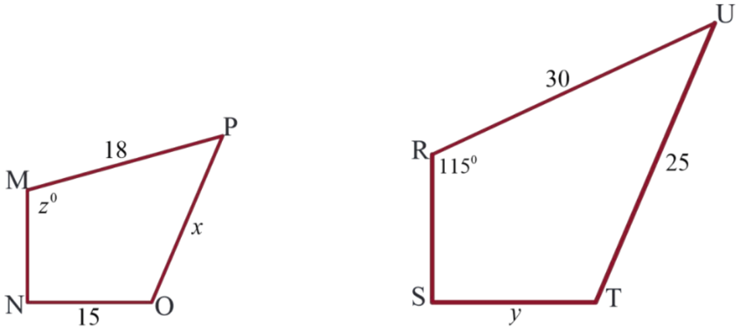
Solution:

$$\frac{DE}{AB} = \frac{8}{4} = 2\,, \quad \frac{EF}{BC} = \frac{10}{5} = 2, \quad \frac{FD}{CA} = \frac{4}{2} = 2,$$

Thus, the common ratio is 2.

Example 2

In the following quadrilaterals, assume $PMNO \sim URST$. Find the unknowns x, y, z



In the similarity statement, $\angle M = \angle R$, So, $z^0 = 115^0$.

To find the unknown lengths x and y set up proportions of the corresponding sides as

$$\frac{PM}{UR} = \frac{PO}{UT}$$

$$\frac{18}{30} = \frac{x}{25}$$

$$x = 15$$

Similarly, to find y,

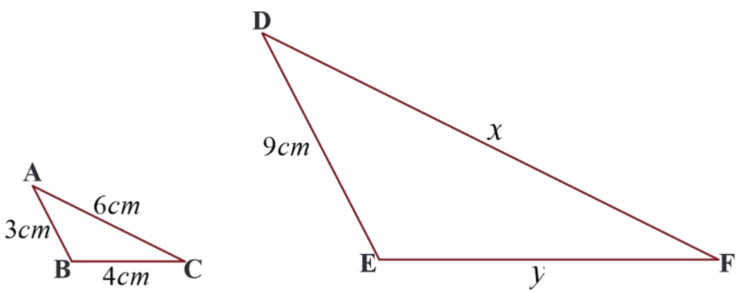
$$\frac{PO}{UT} = \frac{NO}{ST}$$

$$\frac{15}{25} = \frac{15}{y}$$

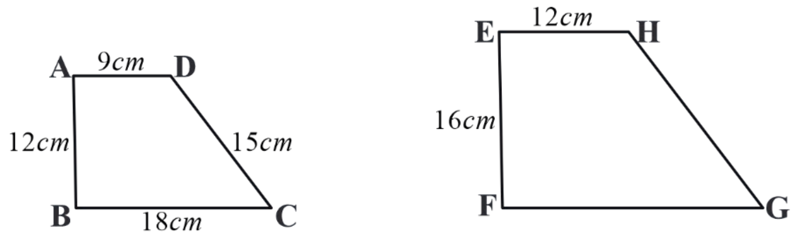
$$y = 25$$

Exercise 7.6

a) Given $\Delta ABC \sim \Delta DEF$. Find x and y.

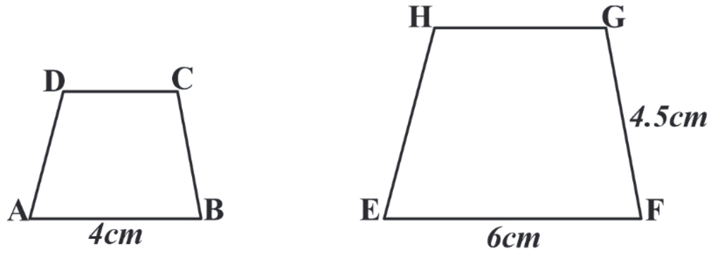


b) Figure $ABCD$ is similar to figure $EFGH$. Find the perimeter of $EFGH$.



c) Given $ABCD \sim EFGH$, find the common ratio of their corresponding sides.

Find BC



Exercise 7.7

For the following questions, determine whether each of the statement is true or false.

a. All equilateral triangles are similar.

b. All isosceles triangles are similar.

c. All isosceles right triangles are similar.

d. All rectangles are similar.

e. All rhombuses are similar.

f. All squares are similar.

g. All congruent polygons are similar.

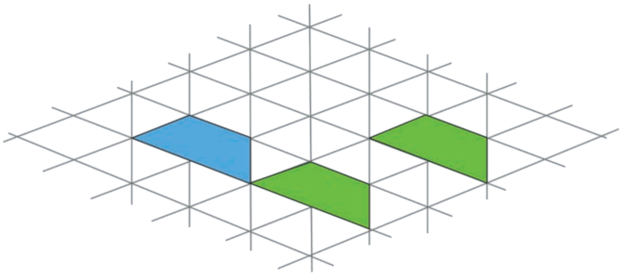
h. All similar polygons are congruent.

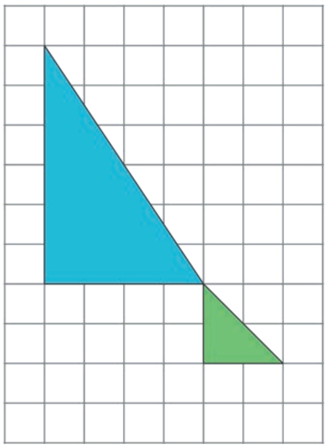
i. All regular pentagons are similar.

7.3 Theorems on similar plane figures

Activity 7.3

Determine whether the following figures are similar or not.





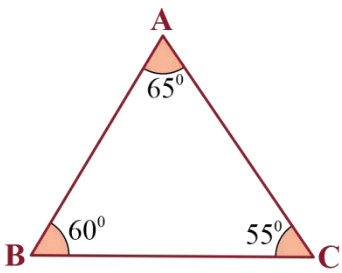
Simply as two people can look at a painting and notice or sense differently the piece of the art work, there is always more than one way to create a proper proportion given similar triangles. These theorems are very useful in such a way that we can conclude the similarity of a pair of triangles by showing only three pair of sides or /and angles among the six pairs of congruent angles and sides.

For example, we can say: The three theorems: Angle - Angle (AA), Side - Side - Side (SSS) and Side - Angle - Side (SAS) are important for determining similarity in triangles.

7.3.1 AA similarity Theorem

The AA similarity theorem for triangles states that if the two angles of one triangle are respectively congruent to the two angles of the other, then the triangles are similar. In short, equiangular triangles are similar. Ideally, the name of this criterion should then be the AAA (Angle-Angle-Angle) similarity theorem, but we call it as AA similarity theorem because we need only two pairs of angles to be equal - the third pair will then automatically be equal by the angle sum property of triangles. The notation ~ is used to denote similarity.

Consider the following figures, in which ΔABC and ΔDEF are equiangular.

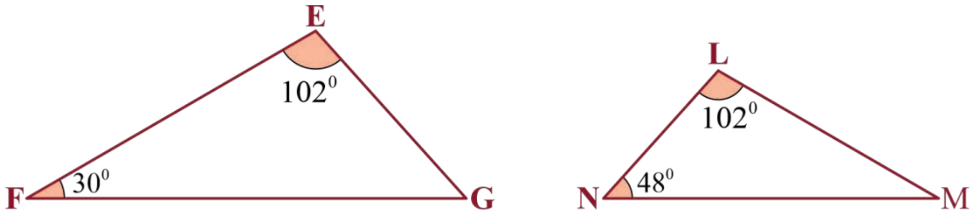


Using the *AA* similarity theorem, we can say that these triangles are similar.

Mathematically, it is written as $\Delta ABC \sim \Delta DEF$.

Example

Determine whether or not the following two triangles are similar.

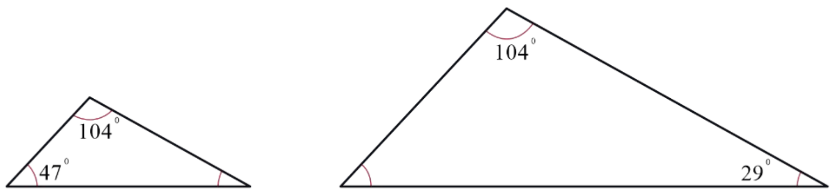


Solution:

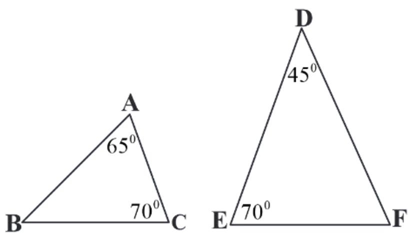
Compare the angles to see if we can use the AA similarity theorem. Using triangle angle sum theorem, $m(\angle G) = 48^o$ & $m(\angle M) = 30^o$. Hence, $\Delta EFG \sim \Delta LMN$ by AA similarity theorem.

Exercise 7.8

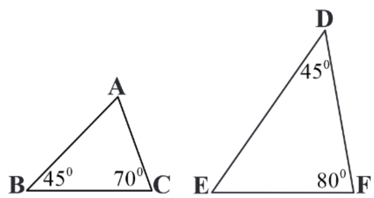
Are the following pairs of triangles similar or not by AA similarity theorem or not? Why?



a



b



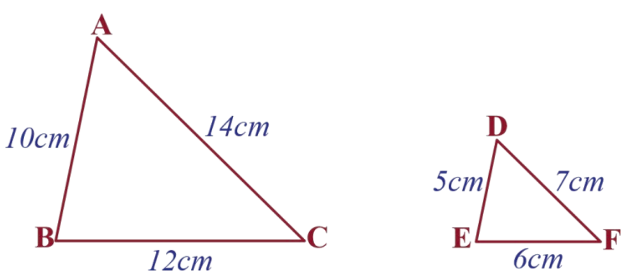
c

7.3.2 SSS similarity Theorem

If the corresponding three sides of two triangles are proportional to each other, then the triangles are similar. This essentially means that any such pair of triangles are equiangular (all corresponding angle pairs are congruent).

Example 1

Determine whether the following pairs of triangles are similar by SSS similarity theorem.



Solution:

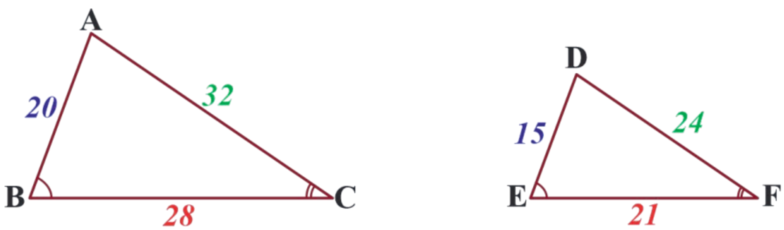
$$\frac{AB}{DE} = \frac{BC}{EF} = \frac{AC}{DF} = 2.$$

Hence, $\Delta ABC \sim \Delta DEF$ by SSS similarity theorem.

Example 2

When the corresponding sides of two triangles are proportional, the triangles are similar. What are the corresponding sides? Using the triangles below, we see how the sides are line up in the diagram.

Solution:

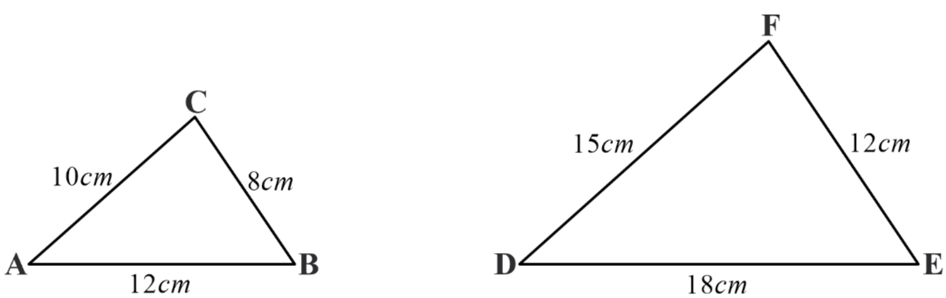


Since $\frac{28}{21} = \frac{32}{24} = \frac{20}{15} = \frac{4}{3}$ the same color are the corresponding sides.

Therefore, the triangles are similar by SSS similarity theorem.

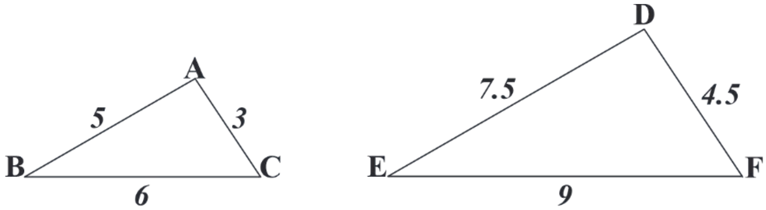
Exercise 7.9

a) Determine whether the following pairs of triangles are similar by SSS similarity theorem or not.

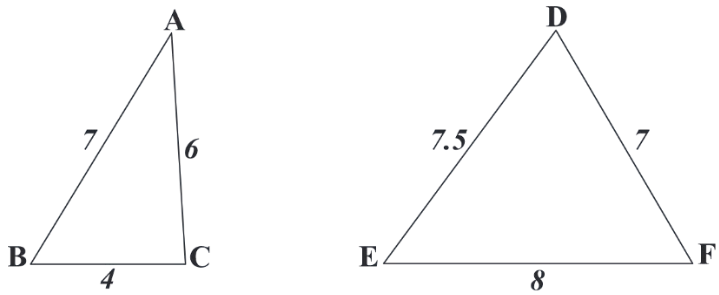


b) Determine whether the following pairs of triangles are similar.

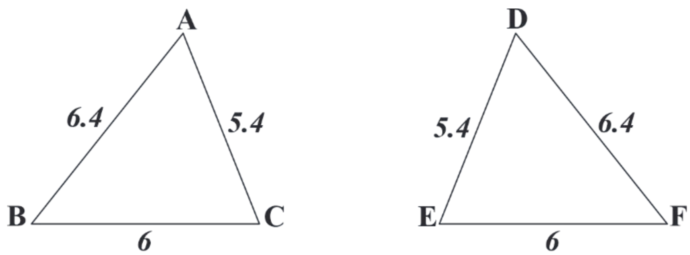
(1)



(2)



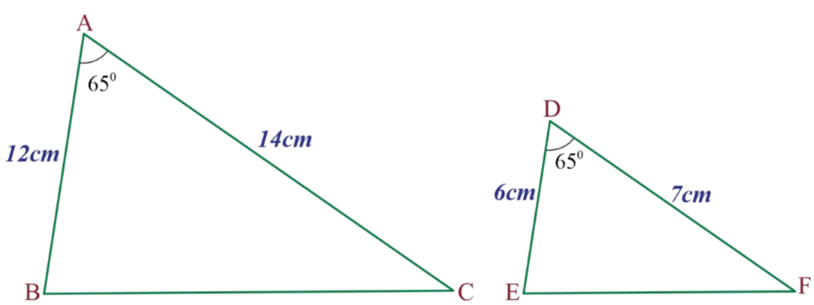
(3)



7.3.3 SAS similarity theorem

If the corresponding two sides of two triangles are proportional and the included angles are congruent, then the two triangles are similar.

Note the emphasis on the word included. If the congruent angles are non-included angles, then the two triangles may not be similar. Consider the following figures.



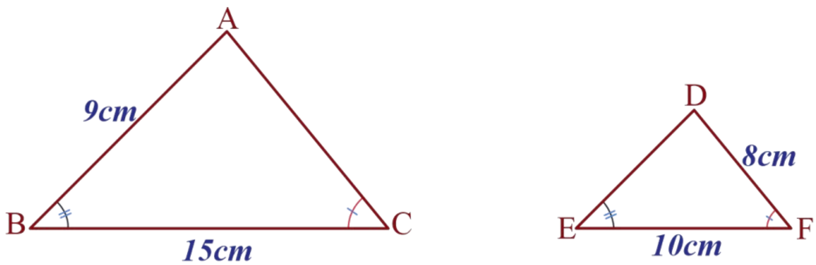
$\frac{AB}{DE} = \frac{12}{6} = 2$, $\frac{AC}{DF} = \frac{14}{7} = 2$, then the two corresponding sides are in the same ratio and $\angle A \equiv \angle D$

Hence, $\Delta ABC \sim \Delta DEF$ by SAS similarity theorem.

Example 1

Assume $\Delta ABC \sim \Delta DEF$

Find the length of the remaining sides of the triangles.



Solution:

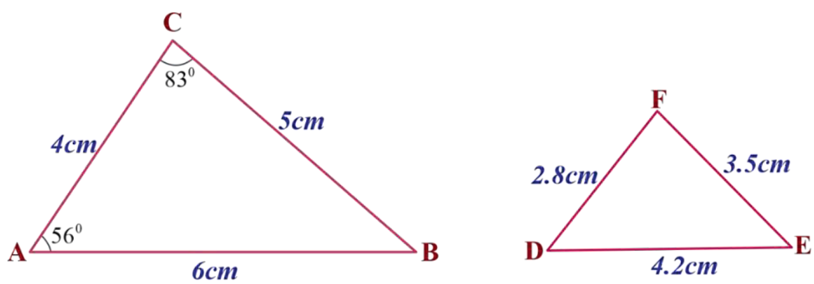
Since the triangles are similar, we have relation

$$\frac{AB}{DE} = \frac{BC}{EF} = \frac{AC}{DF}.$$

Here, $\frac{9}{DE} = \frac{15}{10}$. This implies $DE = 6cm$. Also, $\frac{9}{6} = \frac{AC}{8cm}$, gives $AC = 12cm$.

Example 2

Consider the following figures and find the value of $\angle E$.



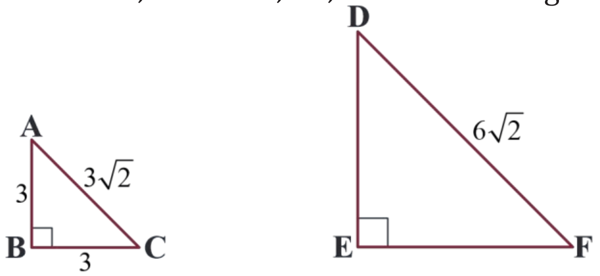
Solution:

Match the longest side with the longest side and the shortest side with the shortest side and check all the three ratios. We note that the three sides of the two triangles are respectively proportional, that is $\frac{6}{4.2} = \frac{5}{3.5} = \frac{4}{2.8} = \frac{10}{7}$. Therefore, $\Delta ABC \sim \Delta DEF$

by SSS similarity theorem. Hence, the corresponding angles are congruent and hence $\angle E = \angle B = 180^0 - (56^0 + 83^0) = 41^0$.

Example 3

Given $\Delta ABC \sim \Delta\ DEF$, Find $\angle D, \angle E, \angle F$ and the lengths of DE .



Solution:

As $\Delta ABC \sim \Delta\ DEF$, $\angle D = \angle A = 45°$ since the triangles are isosceles triangles.

$\angle E = \angle B = 90°$,

$\angle F = \angle C = 45°$,

$$\frac{DE}{AB} = \frac{DF}{AC}$$

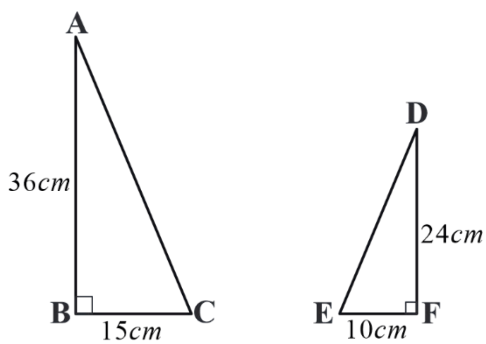
$$\frac{DE}{3} = \frac{6\sqrt{2}}{3\sqrt{2}}$$

$$\frac{DE}{3} = 2$$

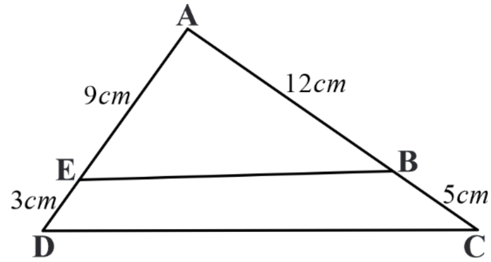
$$DE = 6$$

Exercise 7.10 a)

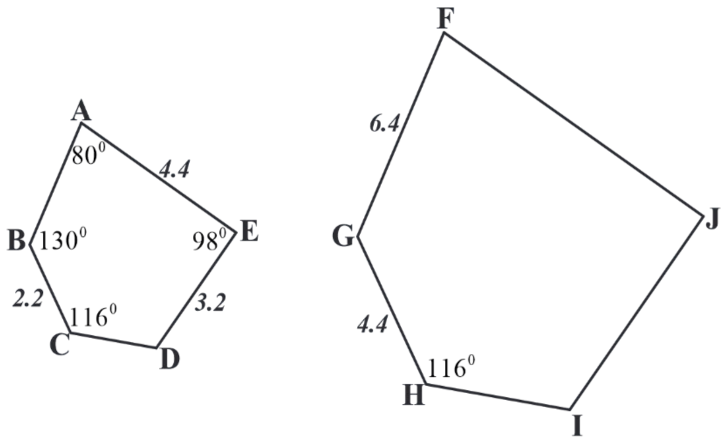
a. Are these two triangles similar? Why?



b. Are there similar triangles in the figure below? Why?



c. Given ABCDE ~FGHIJ, Find ∠D, ∠F, ∠J and ∠I and the lengths of IJ and JF.



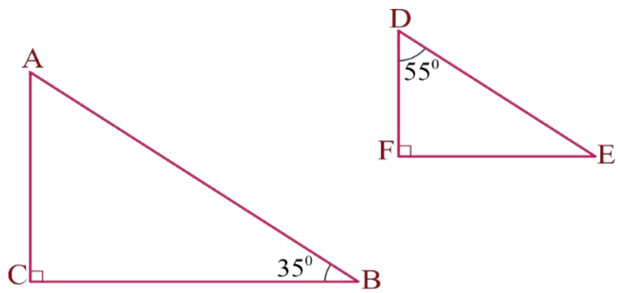
Exercise 7.10 b)

1. Answer the following questions referring the triangles below:

   Given $AB = 20$ and $DE = 15$.

   i. Are these triangles similar?

   ii. Find BC if $FE = 12$.



2. What do we mean when we say two triangles are congruent? Two triangles are similar? What is the difference between congruency and similarity?

3. If $\Delta BIG \sim \Delta HAT$, then

a. list the congruent angles and proportions of the sides.

b. If $BI = 9$ and $HA = 15$, find the scale factor.

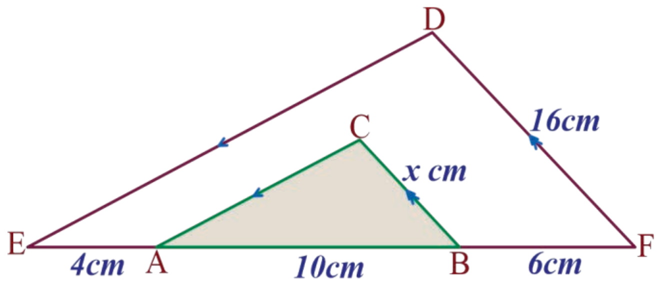
c. If $BG = 21$, find HT. (Use the scale factor above)

d. If $AT = 45$, find IG.

4. As shown in the figure below, DE is parallel to CA and DF is parallel to CB. Answer the following questions:

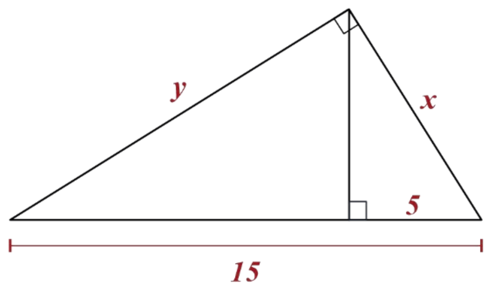
a. Show, $\Delta ABC \sim \Delta EFD$

b. Find the value of x.

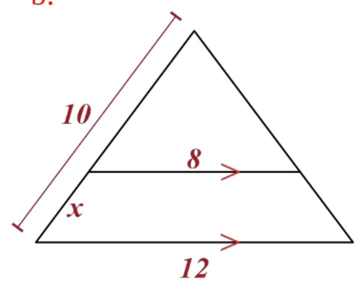


5. Find the unknown numbers *x*, *y* and *z* of the following figures.

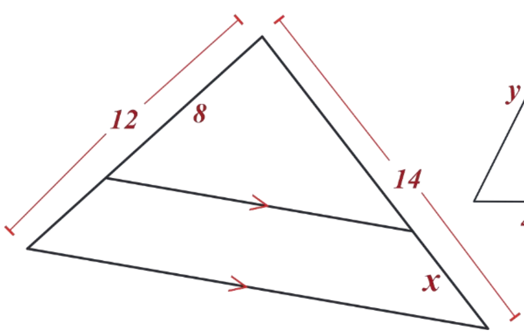
a.



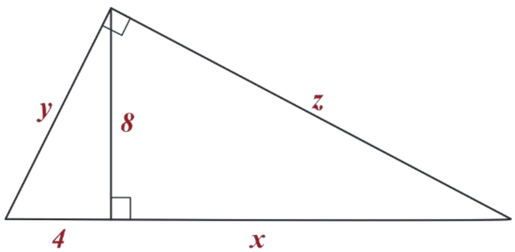
b.



c.



d.

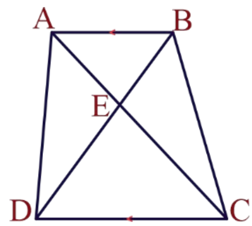


6. Suppose $\overline{AB}$ and $\overline{DC}$ are parallele in the figure below. Use congruency of alternate interior angles.

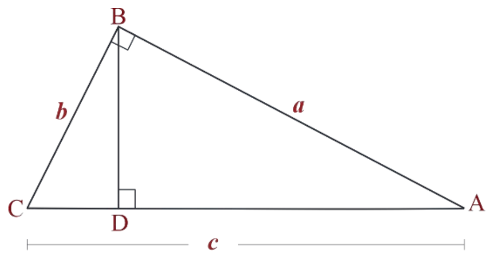
a. Mention two similar triangles. How you check they are similar?

b. Can you mention triangles which are not similar?

c. If $AB = 10, AE = 7$ and $DC = 22,$ then find AC.



7. Prove Pythagoras theorem using similarity of triangles. (Hint: Consider a right angle triangle, ΔABC and draw segment BD perpendicular to AC. The three triangles are similar.)



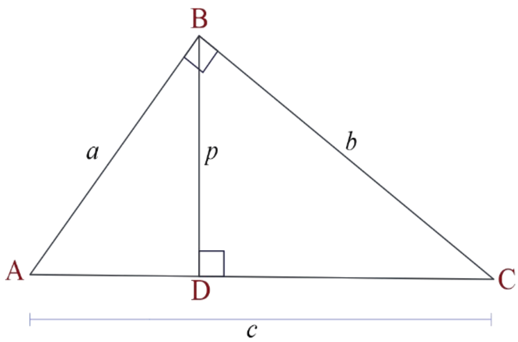
8. ABC is a right angle triangle, right angled at B. If BD, the length of the perpendicular from B on AC, is p,

$BC = b, AC = c$ and $AB = a$

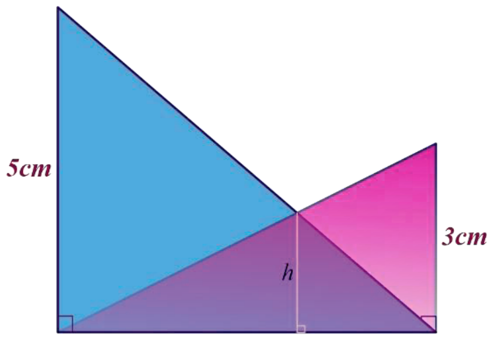
($the\ figure\ below$), show that:

i. $pc = ab$

ii. $\frac{1}{p^2} = \frac{1}{a^2} + \frac{1}{b^2}$



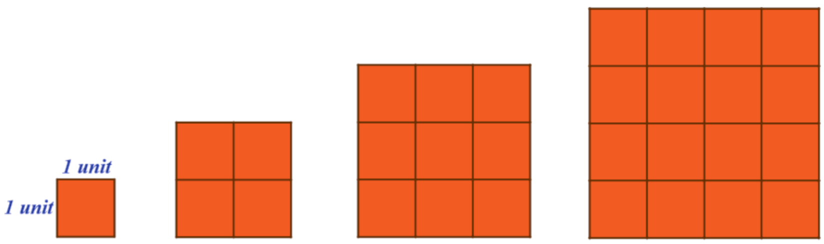
9. Find the perpendicular distance from the point of intersection E of the hypotenuses of the right angled triangles to their common base. Is there any other method to solve this problem? What can you conclude?



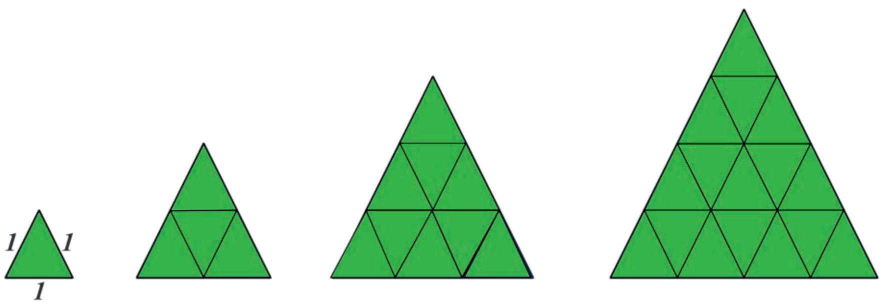
7.4 Ratio of perimeters of similar plane figures

Activity 7.4

1. What is ratio?
2. What is a scale factor?
3. Use pattern blocks to make a figure whose dimensions are 1, 2, 3 and 4 times larger than the original figure. Find the perimeter P of each larger figure.



$P = 4$ $P = 8$ $P = 12$ $P = 16$



$P = 3$	$P = 6$	$P =$ ☐	$P =$ ☐
$A = a$	$A = 4a$	$A =$ ☐	$A =$ ☐

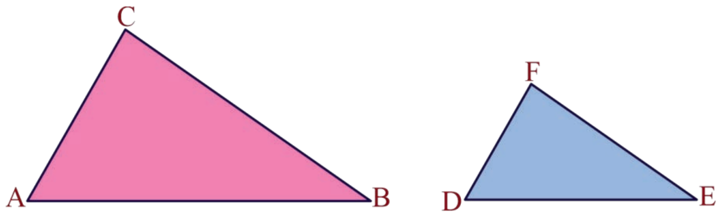
4. How do changes in dimensions of similar figures affect the perimeters of the figures? Is this always so? What can you generalize?

Definition 7.3

If two figures are similar, then the ratio of their perimeters is equal to ratio of their corresponding side lengths.

Example 1

Assume that $\Delta ABC \sim \Delta DEF$ and the ratio of the lengths of their sides is 2. Then, find the ratio of the corresponding perimeters. The perimeter of ΔABC is $6 + 8 + 10$ and the perimeter of ΔDEF is $3 + 4 + 5$.



Solution:

The perimeter of $\Delta ABC = 6 + 8 + 10 = 24$ and

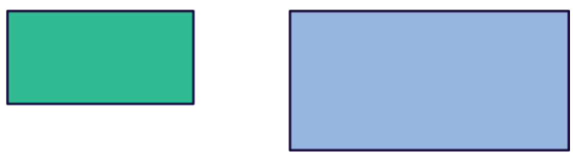
The perimeter of $\Delta DEF = 3 + 4 + 5 = 12$.

$\frac{\text{Perimeter of } \Delta ABC}{\text{Perimeter of } \Delta DEF} = \frac{24}{12} = 2$ is equal to the ratio of the corresponding sides $\left(\frac{AB}{DE} = \frac{BC}{EF} = \frac{AC}{DF}\right)$

Note: $\frac{AB}{DE}$ is the same as AB: DE. Both refer to the ratio of AB to DE.

Example 2

Find the ratio (purple to blue) of the perimeters of the similar rectangles below if the lengths of the rectangles are 4cm and 6cm for purple and blue, respectively.



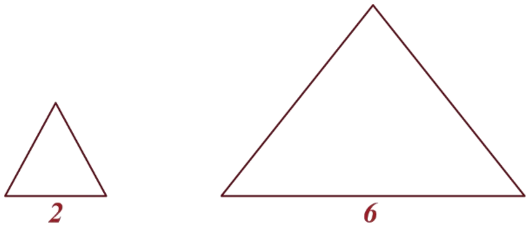
Solution:

$$\frac{\text{Perimeter of the Purple rectangle}}{\text{Perimeter of the Blue rectangle}} = \frac{4cm}{6cm} = \frac{2}{3}$$

Exercise 7.11

Use the formula $\frac{p_1}{p_2} = \frac{s_1}{s_2}$, for two similar plane figures where p_1 and p_2 are the perimeters and s_1 and s_2 are the corresponding sides of the figures

1. Find the ratio of the perimeter of the similar triangles below.



2. The ratio of the perimeters of two similar triangles is $\frac{3}{5}$.

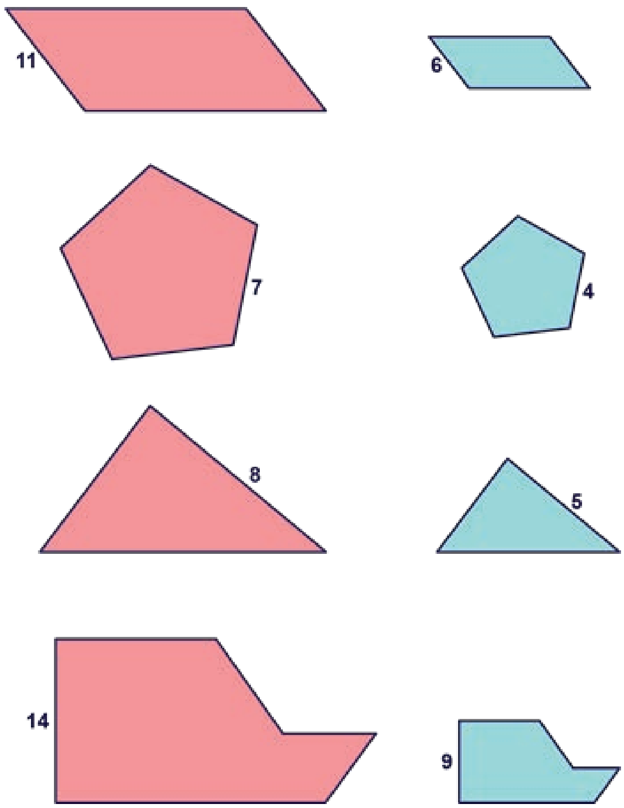
   The scale factor is ________

3. Two circles have radii 8 cm and 12 cm, respectively. The ratio of their circumferences is ______

4. How does doubling the side lengths of a rectangle affect its perimeter?

5. How does quadrupling the side lengths of a rectangle affect its perimeter?

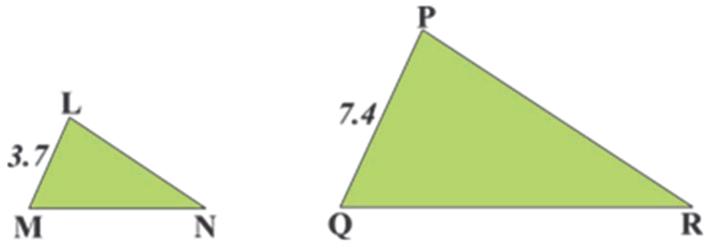
6. The pairs of figures below are similar. Find the ratio of their perimeters (purple to blue)



7.5 Ratio of areas of similar plane figures

Activity 7.5

1. Repeat activity 7.4 question number 3 and 4 for comparing areas of similar plane figures.
2. The two triangles below are similar. Drag any orange dot (vertex) at P, Q, R. The areas of the triangles are 9.2 and 36.8. Verify that the square of the ratio of the two corresponding sides is equal to the ratio of the areas.

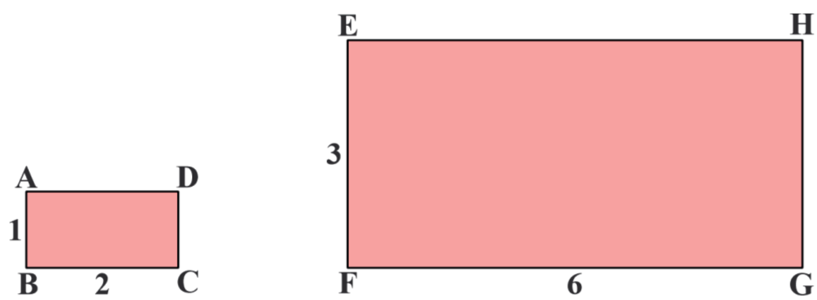


Definition 7.4

If two plane figures are similar, then the ratio of their areas is equal to the square of the ratio of their corresponding side lengths.

Example 1

Consider a pair of similar rectangles 1cm by 2cm and 3cm by 6cm below. The ratio of the corresponding sides is 3. Find the ratio of their areas.



Area of the smaller rectangle $= 1 \times 2 = 2cm^2$

Area of the larger rectangle $= 3 \times 6 = 18cm^2$

The ratio of area $= \frac{Area\ of\ the\ larger\ rectangle}{Area\ of\ the\ smaller\ rectangle}$

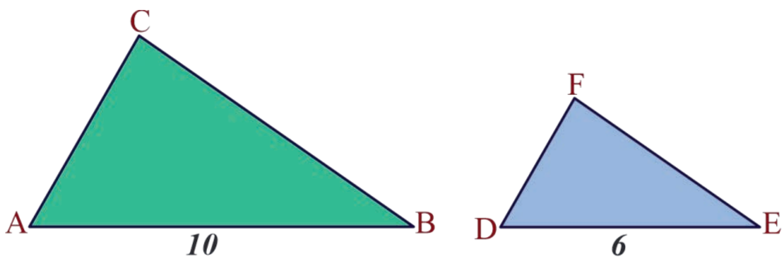
$= \frac{18}{2}$

$= 9 = 3^2 =$ (the ratio of the length of thier corresponding side)^2

In other words, since the ratio of length $= 3:1$, then ratio of area $= 3^2:1^2 = 9:1$

Example 2

If ΔABC and ΔDEF are similar as indicated below. Find the ratio of the area of the larger triangle to the smaller triangle.



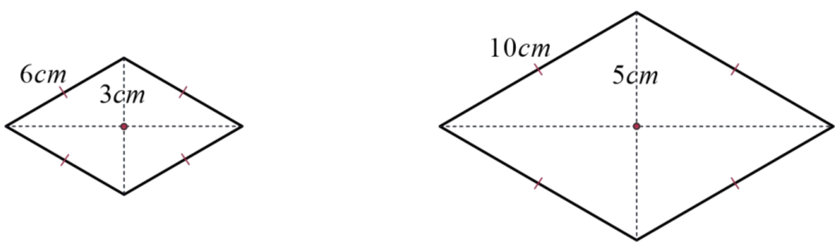
Solution:

The ratio of area $= \frac{\text{Area of the larger triangle}}{\text{Area of the smaller triangle}} = \left(\frac{10}{6}\right)^2 = \frac{25}{9}$.

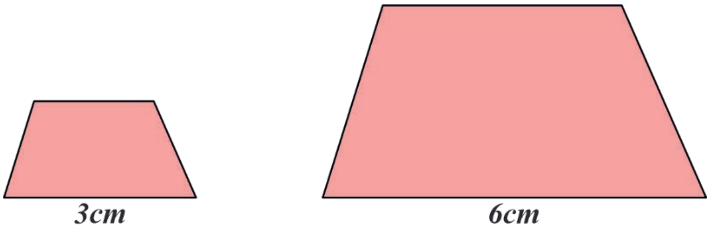
Therefore, the ratio of the areas is $\frac{25}{9}$

Exercise 7.12

a. Find the ratio of the areas of rhombi below. The rhombi are similar.



b. Find the ratio of areas of similar quadrilaterals below.



c. The ratio of the areas of two squares is $64:49$. The scale factor is ____________ and the ratio of their perimeters is_____________.

d. Two circles have radii 8 cm and 12 cm, respectively. The ratio of their circumferences is ____________ and the ratio of their areas is ____________.
(Hint: All circles are similar. The scale factor is the ratio of their radii.)

e. The ratio of the areas of two similar rectangles is $25:36$. The scale factor is ____________ and the ratio of their perimeters is ____________.

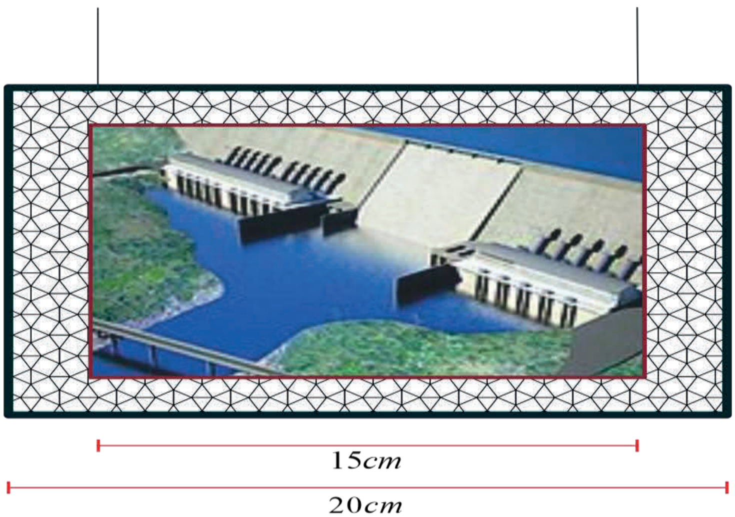
f. Two similar triangles have corresponding sides 3 and 9. If the area of the first triangle is 12, what is the area of the second triangle?

g. Two similar triangles have areas 144 cm^2 and 169 cm^2. If a side of the larger triangle is 26cm, find the length of the corresponding side of the smaller triangle.

h. How does doubling the side lengths of a rectangle affect its area?

i. How does quadrupling the side lengths of a rectangle affect its area?

j. You place a picture on a page. The page and the picture are similar rectangles.



1. How many times greater is the area of the page than the area of the picture?
2. If the area of the picture is 45 cm^2, what is the area of the page?

k. A triangle with area of $10m^2$ has a base of $4m$. A similar triangle has an area of $90m^2$. What is the height of the larger triangle?

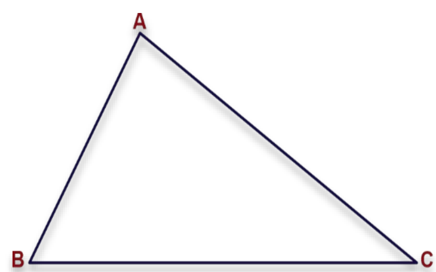
7.6 Construction of similar figures

Activity 7.6

1. What is a scale factor?
2. Given a line segment, how do you draw a line segment parallel to the given lines segment?
3. What does it mean that the scale factor is greater than 1? Less than1? Equal to 1?

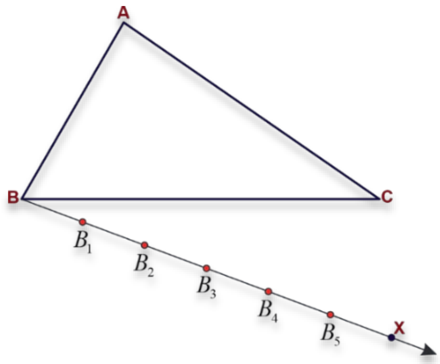
The construction of a similar triangle involves two different cases. On one hand, the triangle to be constructed could be bigger (or larger), and on the other hand, the triangle may be smaller than the given triangle. Also, the scale factor determines the ratio of the sides of the triangle to be constructed with the corresponding sides of the given triangle.

Example: construct a triangle similar to the given triangle ABC for example, with **scale factor** $\frac{5}{3}$.

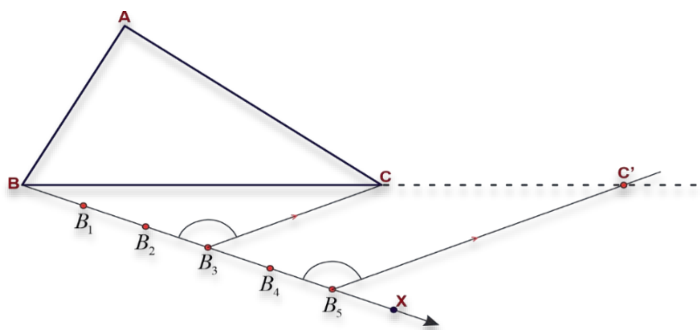


Steps of construction:

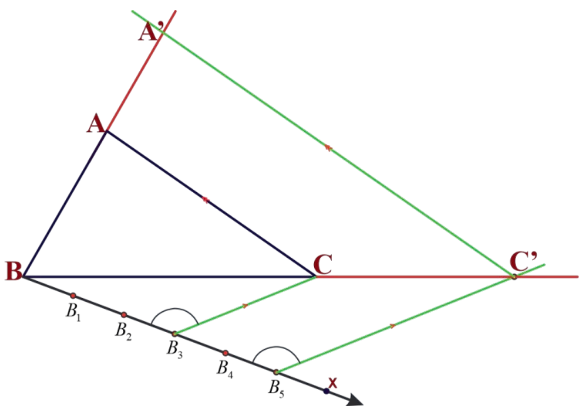
Step 1: Draw a ray BX making an acute, acute with the base BC and mark 5 points $B_1,\ B_2,\ B_3,\ B_4, B_5$ on BX such that $BB_1 = B_1B_2 = B_2B_3 = B_3B_4 = B_4B_5$.



Step 2: Join B_3C and draw a line B_5C' such that B_3C is parallel to B_5C' , where C' lie on the produced BC.



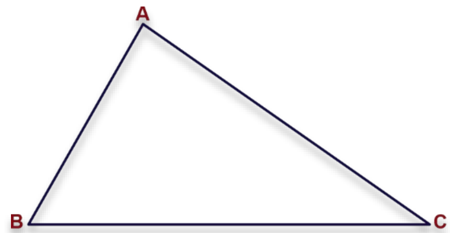
Step 3: Now draw another line parallel to AC at C' such that it meets the produced BA at A'.



Hence, $\Delta A'B'C'$ is the required triangle similar to the ΔABC.

Exercise 7.13

Construct a triangle similar to the given triangle ABC, with scale factor 1, $\frac{1}{2}$ and $\frac{2}{3}$.



7.7 Applications of similarity

Activity 7.7

Given a ladder $5m$ long lying on a wall. If the height of the wall to the top of the ladder is 3m. How far is the bottom of the ladder from the wall?

Similar triangles are used to work out the heights of tall objects such as trees, buildings and towers without climbing, which are difficult to measure for us.

Example 1

Almaz is 1.6 meters tall and is standing outside next to her younger brother. She notices that she can see both of their shadows and decides to measure each shadow. Her shadow is 2 meters long and her brother's shadow is 1.5 meters long. How tall is Almaz's brother?

Solution:

Let x be the height of the younger brother. Using proportionality of corresponding heights and length of shadows, we have

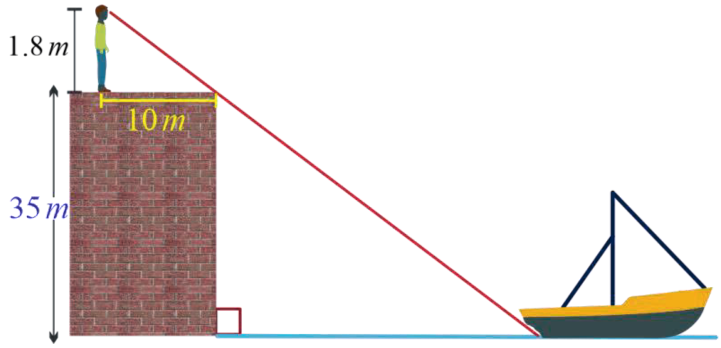
$\frac{1.5}{2} = \frac{x}{1.6}$, Hence, $x = 1.2$ meters.

Therefore, Almaz's brother is $1.2m$ long.

Example 2

Abebe's boat has come untied and floated away on Lake Tana. He is standing on the top of a building that is 35 m above the water from the lake. If he stands 10 m from the edge of the building, he can visually align the top of the building with the water at the back of his boat. His eye level is 1.8 m above the top of the building. Approximately, how far is Abebe's boat from the building?

Solution:



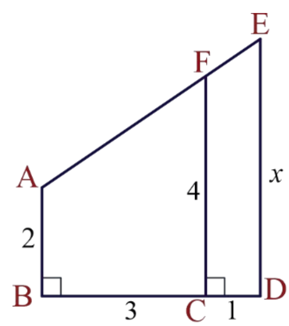
Let x be the distance from the boat to the building. Using proportionality of corresponding sides of the two triangles, we have

$\frac{35}{1.8} = \frac{x}{10}$, Hence, $x \sim 194.4$m.

Therefore, the boat is approximately 194.4m from the building.

Example 3

Suppose BA, CF and DE be the heights of three buildings. Let $DC = 1, BC = 3$, $CF = 4$ and $AB = 2$ as indicated in the figure below show. Find the height x of the tallest building.



Solution:

Extend FA and CB so that they meet at G. Now, we have ΔDGE~ΔCGF by AA similarity theorem.

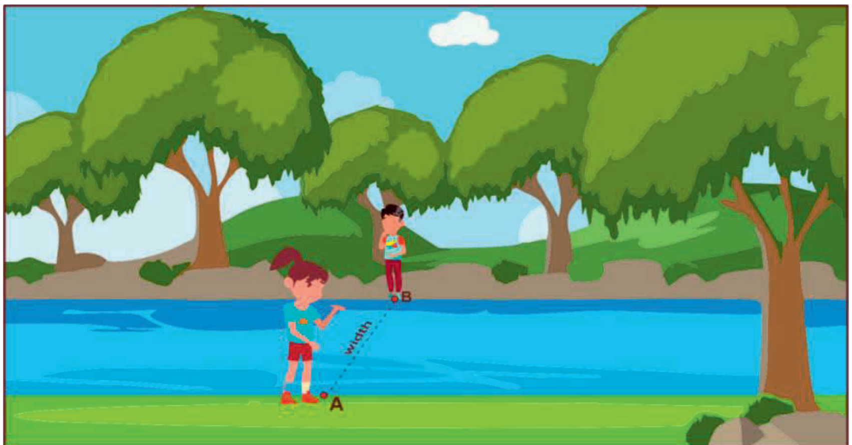
$\frac{\text{DG}}{\text{CG}} = \frac{\text{ED}}{\text{FC}}$. Hence, $\frac{\text{GB}+4}{\text{GB}+3} = \frac{x}{4}$. Now, consider similar triangles AGB and FGC. From this, we get $\frac{\text{GB}}{\text{GC}} = \frac{\text{AB}}{\text{FC}}$.

$\frac{GB}{GB+3} = \frac{2}{4}$. Therefore, $GB = 3$ and $x = \frac{14}{3}$

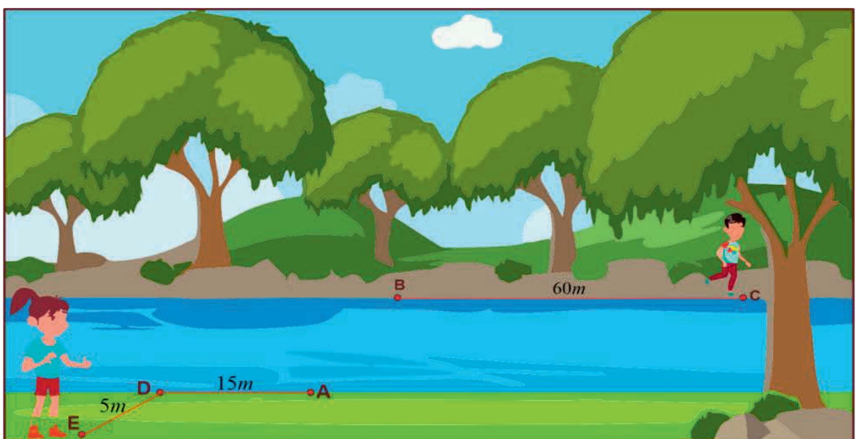
Example 4

Two persons stand at point A and B and want indirectly to measure the width of the river. How long is the width of the river? It is done as follows: Person A moves 15 m to the left and person B moves 60 m to the right. Moreover, person A moves 5m away from the river. The line connecting the places after the two people have moved passes through point A.

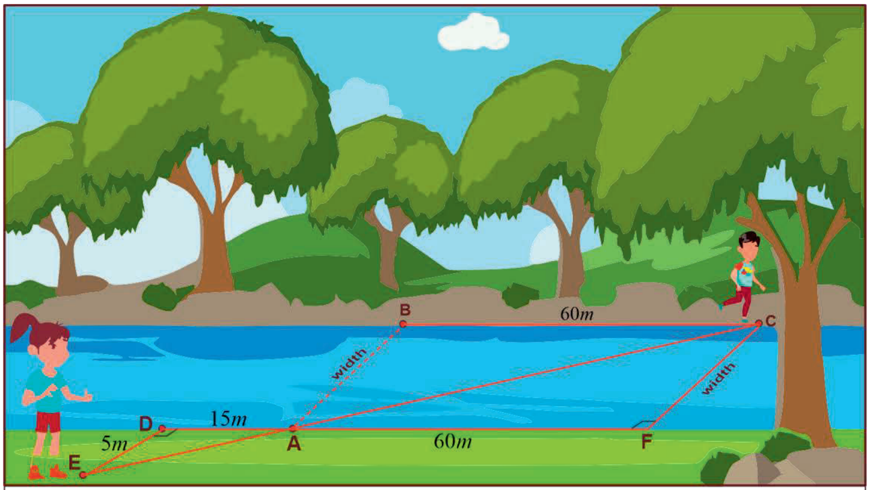
Solution:



Their Original Position



Person at B moves the right 60m at C
and, Person at A moves to left 15m at D and then 5m away from the river at E



Similar Triangles ADE and AFC are formed

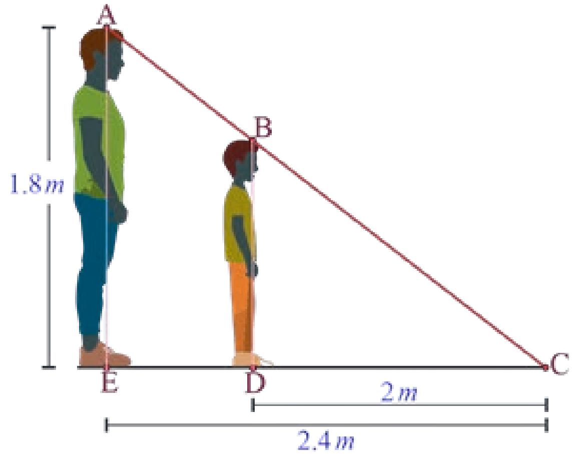
Therefore, $\frac{AD}{AF} = \frac{DE}{FC} = \frac{AE}{AC}$

$\frac{15}{60} = \frac{5}{FC} = \frac{AE}{AC}$.

$FC = 20$. Hence, the width of the river is $20m$

Exercise 7.14

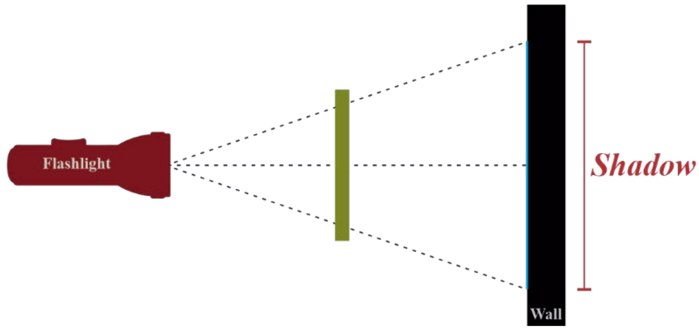
a. Abdi is 1.8 meters tall and is standing outside next to his younger brother. He notices that he can see both of their shadows and decides to measure each shadow. His shadow is 2.4 meters long and his brother's shadow is 2 meters long. How tall is Abdi's brother?



b. Abdisa's boat has come untied and floated away on Lake Hawasa. He is standing on the top of a building that is 75 m above the water from the lake. If he stands 15 m from the edge of the building, he can visually align the top of the building with the water at the back of his boat. His eye level is 1.75 m above the top of the building. Approximately, how far is Abebe's boat from the building?

Exercise 7.15

1. A giraffe is 6m tall and casts a shadow of 4m. Another small giraffe casts a shadow of 2m. How long is the smaller giraffe?
2. A flagpole casts a shadow 9m long. A person 1.8 m long, standing nearby casts a shadow 4m long. How tall is the flagpole?
3. A photograph measuring 4cm wide and 5cm long is enlarged to make a wall mural. If the mural is 120cm wide, how long is the mural?
4. A 9m ladder leans against a building 6m above the ground. At what height would a 15m ladder touch the building if both ladders form the same angle from the ground?
5. A ladder is placed against a wall such that it aches up to a height of 4m of the wall. If the foot of the ladder is 3m away from the wall, find the length of the ladder.
6. P and Q are points on the sides CA and CB, respectively of ΔABC, right angled at C. Prove that $AQ^2 + BP^2 = AB^2 + PQ^2$.
7. A $12 - centimeter$ rod is held between a flashlight and a wall as shown. Find the length of the shadow on the wall if the rod is 24 cm from the wall and 15 cm from the light. Assume that the dotted line segments meet at a point inside the flashlight.


Flashlight
Shadow
Wall

Summary

1. Two plane figures are similar if:

   i. their corresponding sides are proportional.

   ii. their corresponding angles are congruent.

2. Under an enlargement

   a. Lines and their images are parallel.

   b. Angles remain the same.

   c. All lengths are increased or decreased in the same ratio.

3. Scale factor- the ratio of corresponding sides usually expressed numerically so that:

$$\text{Scale factor} = \frac{\text{length of the line segment on the enlagement}}{\text{length of the line segment on the original}}$$

4. AA Similarity theorem

   If two angles of one triangle are congruent to the corresponding two angles of another triangle, then the two triangles are similar.

   SSS Similarity theorem

   If the corresponding three sides of two triangles are proportional to each other, then the triangles are similar.

   SAS Similarity theorem

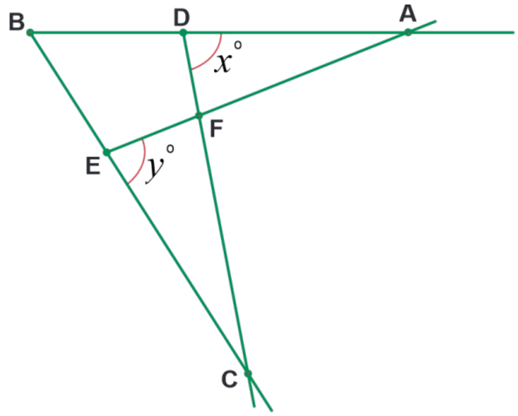
   If the corresponding two sides of two triangles is proportional and the included angles are congruent, then the two triangles are similar.

5. If two figures are similar, then the ratio of their perimeters is equal to ratio of their corresponding side lengths.

6. If two figures are similar, then the ratio of their areas is equal to the square of the ratio of their corresponding side lengths.

Review Exercise

1. In the given figure, if $x = y$ and AB = CB, then prove that AE = CD.



2. AD is an altitude of an isosceles ΔABC in which $AB \equiv AC$.

Show that

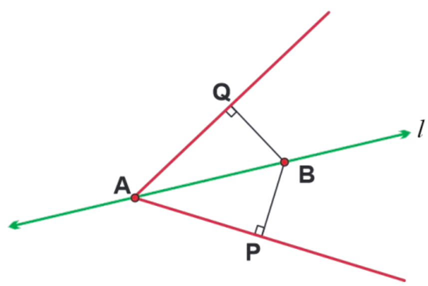
(i) AD bisects

(ii) AD bisects $\angle A$.

3. In the given figure, line l is the bisector of $\angle A$ and B is any point on l. If segment BP and BQ are perpendiculars from point B to the arms of $\angle A$, show that

(i) $\Delta APB \equiv \Delta AQB$

(ii) $BP \equiv BQ$, i.e., B is equidistant from the arms of $\angle A$.

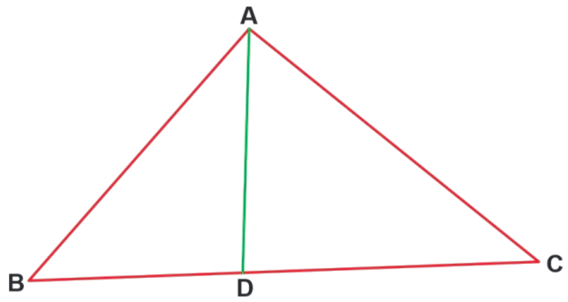


4. In a triangle, a line is drawn parallel to one side and a small triangle is cut off. Prove that the triangle cut off is similar to the original triangle.

5. Consider the four triangles made by joining all the midpoints of a triangle. Show these triangles are congruent and they are similar to the largest triangle.

6. Prove that a parallelogram is formed by joining the midpoints of any quadrilateral,

7. Assume that D is a point on side BC of triangle ABC such that $\angle ADC = \angle BAC$ as in the figure. Show that $(CA)^2 = \mathrm{CB} \cdot \mathrm{CD}$.



8. L and M are the mid-points of the sides AB and AC of ΔABC, right angled at B. Show that $4(\mathrm{LC})^2 = (\mathrm{AB})^2 + 4(\mathrm{BC})^2$.

9. The perimeter of two similar triangles ABC and DEF are 12 cm and 18 cm. Find the ratio of the area of ΔABC to that of ΔDEF

10. The altitudes AD and PS of two similar triangles ABC and PQR are of length 2.5 cm and 3.5 cm. Find the ratio of area of ΔABC to that of ΔPQR.

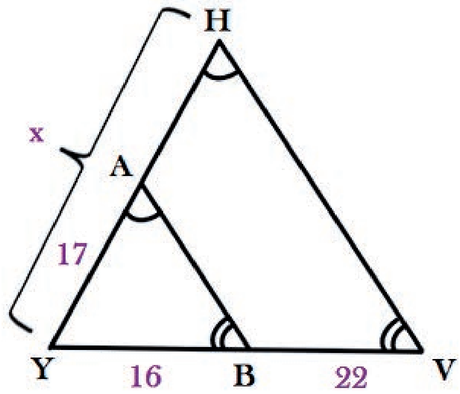
11. Two poles of heights 12 m and 17 m, stand on a plane ground and the distance between their feet is 12 m. Find the distance between their top.

12. A ladder is placed against a wall and its top reaches a point at a height of 8 m from the ground. If the distance between the wall and foot of the ladder is 6 m, find the length of the ladder.

13. In an equilateral triangle, show that three times the square of a side equals four times the square of a median.

14. A triangle with an area of 60 square meters has a base of 4 meters. A similar triangle has an area of 50 square meters. Find the height of the smaller triangle.

15. The value of x in the diagram below is:



A. 38 B. 33.375 C. 40.375 D. 39.75

16. If $\Delta ABC \sim \Delta DEF$, then which of the following is not true?

A. $\frac{AB}{DE} = \frac{BC}{EF}$ B. $\frac{AB}{DE} = \frac{EF}{BC}$ C. $\frac{AC}{DF} = \frac{BC}{EF}$ D. $\frac{DE}{AB} = \frac{EF}{BC}$

17. If the ratio of sides of triangles ABC and DEF is k, then the ratio of their perimeters is:

A. k B. k^2 C. $\frac{1}{k}$ D. $k+1$

UNIT 8

VECTORS IN TWO DIMENSIONS

Unit Outcomes

By the end of this unit, you will be able to:

- **Conceptualize vectors in the sense of direction and magnitude symbolically.**
- **Perform operations on vectors.**

Unit Contents

KEY Words

- direction of a vector
- equality of vectors
- parallel vectors
- addition of vectors
- triangle law of vector addition
- parallelogram law of vector addition
- scalar quantities
- subtraction of vectors
- vector quantities
- magnitude of a vector
- position vector

INTRODUCTION

In your day to day activities, you have an experience of measurements of different physical quantities. In the process of measuring physical quantities, you may express what you measure by using real numbers with appropriate unit. For example, the dimension of your class room may be 6 meter by 8 meter, the average speed of a car is 45 km/hr, the area of a football field is 5,000 m^2, the average annual temperature of Dallol is 34.6° C, the amount of water a bottle contains 1.5 Liter, etc. In each of these measurements of quantities, direction is not involved.

On the other hand, there are physical quantities which need the direction with its magnitude. For example: A boy is riding a bike with a velocity of 30 km/hr in the north-east direction. Here if we want to define the velocity, we need two things, that is, the magnitude of the velocity (30 km/hr) and its direction (north-east).

In this unit, you will learn scalar and vector quantities, representation of vectors, addition and subtraction of vectors, scar multiplication of vectors and some application of vectors in two dimensions.

Activity 8.1

a. List at least 6 physical quantities.

b. What common property and difference you observed among them?

8.1 Vector and Scalar Quantities

Definition 8.1

Scalars are quantities that are fully described by a magnitude (or numerical value) alone.

Example 1

The mass of a stone is 10 kg. Here, the mass is represented by a single number (10) with appropriate measuring unit(kg) . No direction is needed to represent mass of an object. Hence, mass is a scalar quantity.

Example 2

'The density of pure water is 1 g/cm^3 '. Density is represented by a single number so that it is a scalar quantity.

Example 3

' Temprature of a room is $27°C$ '. Temprature is a scalar quantity.

Definition 8.2

Vectors are quantities that are fully described by both a magnitude and a direction.

There are many engineering applications where vectors are important. Force, acceleration, velocity, electric and magnetic fields are vector quantities and all are described by vectors.

If a projectile is projected up in the atmosphere, its position is described by a vector. Furthermore, computer software is used to control the position of a robot, and in this case the position is also described by a vector.

Example 4

The force action causes an object to move or speed up. When the object shown in figure 8.1 is moved by applying a force, we achieve different effects depending on the direction of the force.

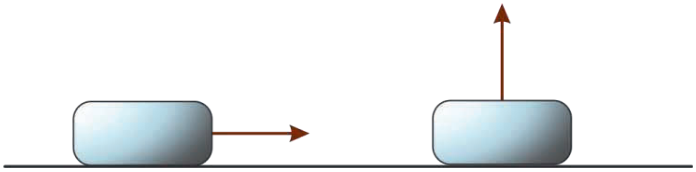


Figure 8.1

In order to specify the force completely, we must state not only its magnitude (its 'strength') but also the direction in which the force acts. For example, we might state that 'a force of 10 Newton's is applied vertically upward (Figure 8.1 of the second part). Clearly this force would have a different effect from one applied horizontally (Figure 8.1 of the first part). The direction in which the force acts is crucial. This indicates that force is a vector quantity.

Example 5

Displacement and *distance* are related quantities which people use interchangeably. However, in more precise language, they are not the same. Distance is a *scalar* whereas displacement is 'directed distance', that is, distance together with a specified direction so that it is a *vector.*

Figure 8.2

So, referring to figure 8.2. Suppose $AB = 20$ meters. We can state that the distance moved is 20 meters, but the displacement is 20 meters in the direction from A to B.

Example 6

Explain the vectors in the following figures

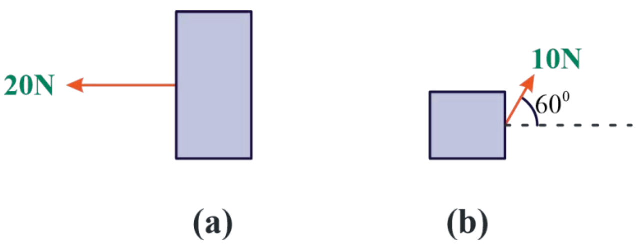


Figure 8.3

Solution:

a. The figure 8.3 (a) shows an object is pulled by a force of 20N horizontally. It is represented by a vector with magnitude 20 N to the left.

b. The figure 8.3 (b) shows an object is pulled by a force of 10 N at an angle of 60^0 with the horizontal.

Exercise 8.1

1. Which statement describes a vector?

   A. It has direction but no magnitude B. It has constant magnitude but no direction

   C. It has magnitude but no direction D. It has both magnitude and direction

2. Scalar quantities are completely described by their____________.

   A. area B. unit C. magnitude D. direction

3. Consider the following quantities and identify whether each is a scalar quantity or a vector quantity.

   i) Amount of rainfall in mm v) The speed of a car

   ii) Temperature in a room vi) Velocity of the boat

   iii) Acceleration of a car vii) The height of a building

   iv) Area of a rectangle viii) The weight of the body

8.2 Representation of a Vector

In this sub unit, we consider the vector represented by a directed line segment on a plane. This is called a vector on a plane.

How do we represent a vector?

A vector can be represented by either algebraically or geometrically. A vector is represented geometrically by a directed line segment (a line segment with direction).

In figure 8.4, the vector is represented by a directed line segment and we denote this by an arrow as $\overrightarrow{AB}$. In this case, the point A is called the *initial* and the point B is called the *terminal*

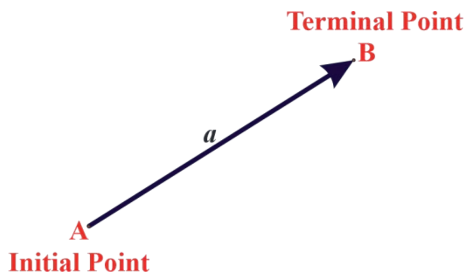


Figure 8.4

A vector also represented by using letters with an arrow bar over it such as $\vec{a}, \vec{u}, \vec{v}$ or a bold small letters like $\boldsymbol{a}, \boldsymbol{u}, \boldsymbol{v}$ etc. Hence, the above vector $\overrightarrow{AB}$ can also be represented by $\vec{u}$.

Equality of vectors

Two vectors are said to be equal if and only if they have the same magnitude and direction.

In figure 8.5, the two vectors have the same magnitude (length) and the same direction.

Hence, they are equal vectors. In this case we can write this as $\overrightarrow{AB} = \overrightarrow{CD}$, or $\boldsymbol{a} = \boldsymbol{b}$.

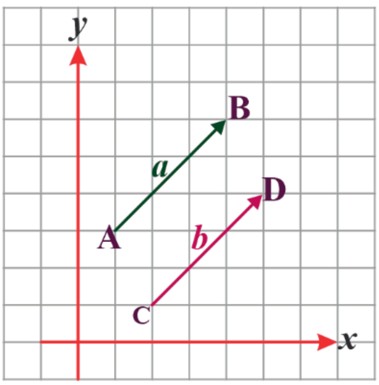


Figure 8.5

Opposite of a vector

The vector which has the same magnitude as that of a vector $\overrightarrow{AB}$, but opposite in direction is called oppositve vector of $\overrightarrow{AB}$ and is denoted by $-\overrightarrow{AB}$. From figure 8.6, you can observe that one vector is the negative of the other. Thus, $\boldsymbol{a} = -(-\boldsymbol{a})$ as shown in the first part of figure 8.6 and $\overrightarrow{AB} = -\overrightarrow{CD}$ as shown in the second part of figure 8.6.

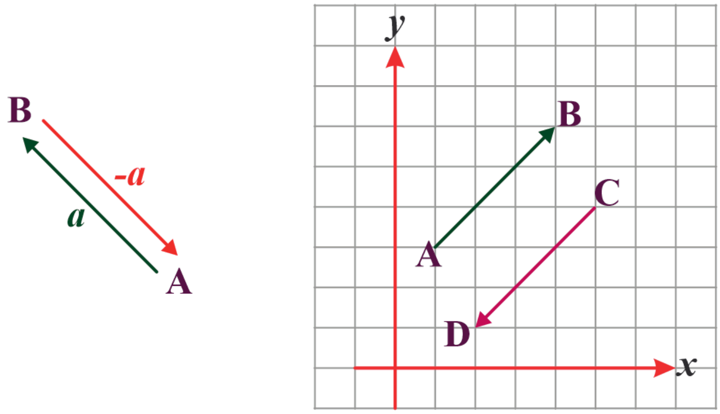


Figure 8.6

Parallel vectors

Vectors that have the same or opposite direction are called parallel vectors.

Example

Identify parallel, equal and opposite vectors from figure 8.7 below.

(a) *(b)*

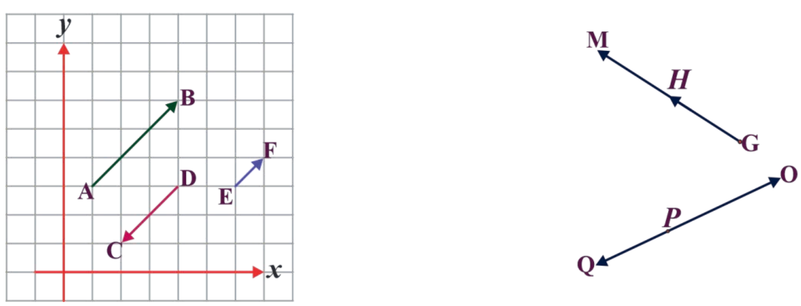


Figure 8.7

Solution::

a. $\overrightarrow{AB}$ and $\overrightarrow{EF}$ are in the same direction, $\overrightarrow{DC}$ is in opposite direction to $\overrightarrow{AB}$ and $\overrightarrow{EF}$.

Hence, these vectors $\overrightarrow{AB}$, $\overrightarrow{EF}$ and $\overrightarrow{DC}$ are parallel to each other.

There is no equal and opposite of a vector in the given figure.

b. $\overrightarrow{PO}$ and $\overrightarrow{PQ}$ are in opposite directions but on the same line, so they are parallel.

Vectors $\overrightarrow{GH}$ and $\overrightarrow{GM}$ are also parallel since they are vectors in the same direction. There is no vector which is opposite of another in the given figure. $\overrightarrow{GH} = \overrightarrow{HM}$ since they have the same direction and equal length(use a ruler to check equality of length).

Exercise 8.2

1. In the figure 8.8, identify the following with reference to $\boldsymbol{a}$.
   a. Equal vector to $\vec{a}$
   b. Opposite to $\vec{a}$
   c. Parallel to $\vec{a}$

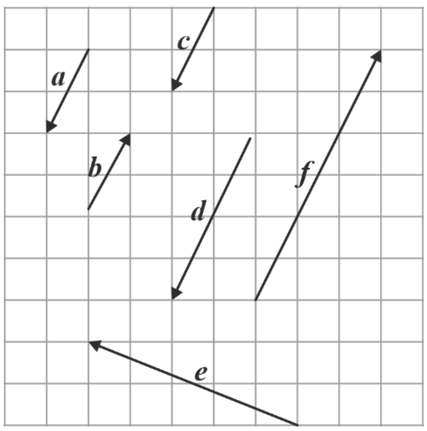


Figure 8.8

2. Write true if the statement is correct and false otherwise.
   a. Two parallel vectors may have common point.
   b. Parallel vectors have the same magnitude.
   c. The magnitude of a vector does not depend on its direction.

Column vector in two dimension

Any vector $\boldsymbol{a}$ can be represented by a column vector $\boldsymbol{a} = \begin{pmatrix} x \\ y \end{pmatrix}$, where x is the horizontal and y is the vertical component of $\boldsymbol{a}$. For example, in the figure 8.9 alongside, the illustrated vector $\begin{pmatrix} 2 \\ 3 \end{pmatrix}$ is denoted by $\boldsymbol{a}$ or $\vec{a}$.

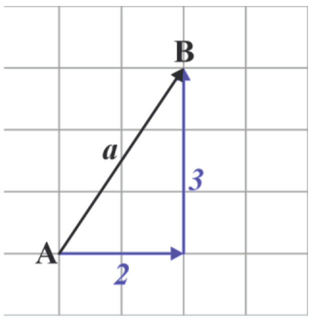


Figure 8.9

We write $\boldsymbol{a} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$ or $\vec{a} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$.

Magnitude of a vector

The magnitude of a given vector $\overrightarrow{AB}$ or $\vec{a}$ as shown in figure 8.9 is the length of the line segment from its initial point A to terminal point B. It is denoted by as $|\overrightarrow{AB}|$ or $|\vec{a}|$.

Example 1

Find the magnitude of the vector $\vec{u} = \binom{4}{3}$.

Solution:

$|\vec{u}|^2 = 4^2 + 3^2$ (Pythagoras theorem)

$= 16 + 9$

$= 25$

Therefore, $|\vec{u}| = |\boldsymbol{u}| = \sqrt{25}$.

Hence, the magnitude of the vector $|\vec{u}|$ is 5 unit.

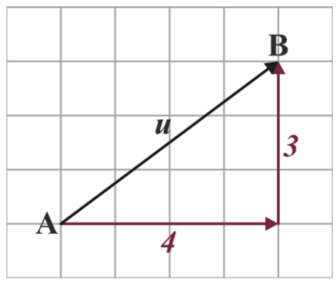


Figure 8.10

Note

- ❖ If a vector is represented on a plane sheet of paper, we can determine its magnitude (length) by measuring its length using a ruler and express it with appropriate unit.
- ❖ Any vector $\boldsymbol{a} = \binom{x}{y}$ has a magnitude (length) $|\boldsymbol{a}| = \sqrt{x^2 + y^2}$.

Example 2

Find the magnitude of each of the following vectors on the coordinate system.

Solution:

How do we get the magnitude of each vector on the coordinate system? We use distance formula. Therefore, $|\overline{PQ}| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$, where $P = (x_1, y_1)$, and $Q = (x_2, y_2)$ are points on the coordinate system. There are three vectors $\overrightarrow{AB}, \overrightarrow{CD}, \overrightarrow{EF}$ in figure 8.11. So, using this distance formula,

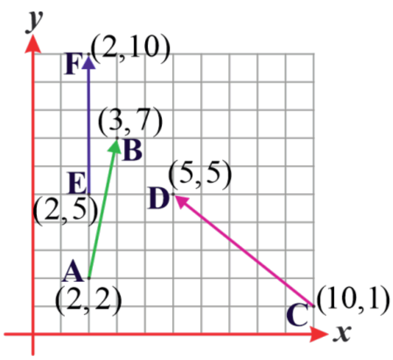


Figure 8.11

$$|\overrightarrow{AB}| = \sqrt{(3-2)^2 + (7-2)^2}$$
$$= \sqrt{1+25}$$
$$= \sqrt{26}.$$
$$|\overrightarrow{CD}| = \sqrt{(5-10)^2 + (5-1)^2}$$
$$= \sqrt{25+16}$$
$$= \sqrt{41}.$$

Similarly, $|\overrightarrow{EF}| = \sqrt{(2-2)^2 + (10-5)^2}$
$$= \sqrt{25}$$
$$= 5\ .$$

Exercise 8.3

1. Answer each of the following questions about vectors $\boldsymbol{a}, \boldsymbol{b}$ and $\boldsymbol{c}$ in the figure 8.12. (Hint: count square grids)
   a. Express each vector in column vector form.
   b. Find the magnitude of each vector.

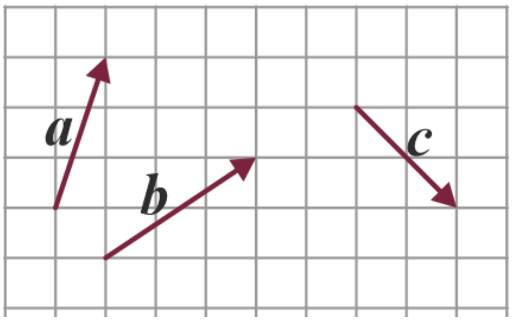


Figure 8.12

2. Find the magnitude of $\boldsymbol{a}, \boldsymbol{b}$ and $\boldsymbol{c}$ in the figure 8.13.

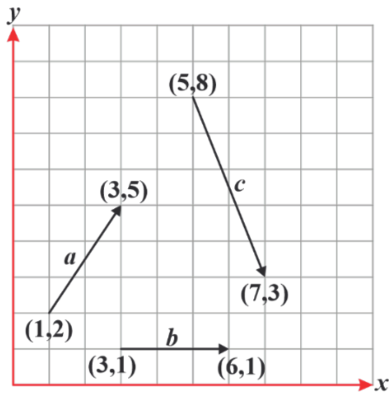


Figure 8.13

Direction of vectors

On a plane, the direction of a vector is given by the angle that the vector makes with a reference direction, often an angle with the horizontal or with the vertical. Expressing vector direction always starts from north/south then angle measure finally east/west.

Example

The direction of the vector $\overrightarrow{AB}$ from the horizontal axis is as shown in figure 8.14.

Let y-axis is North; and x-axis is East, the complement angle of 37° is 53°.

Thus, we can say the direction of B is N 53° E.

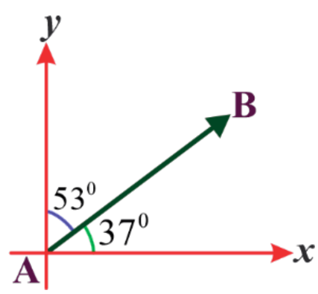


Figure 8.14

Exercise 8.4

1. Locate the vector on the coordinate system whose initial point is $A = (1,1)$ and its terminal point is $B = (5,5)$.
   a. Find the magnitude of $\overrightarrow{AB}$.
   b. Using protractor check that the angle between $\overrightarrow{AB}$ and the horizontal line

passing through A is 45°.

c. What is the direction of $\overrightarrow{AB}$?

2. Locate each of the following on the coordinate system.

   a. $\overrightarrow{AB}$ where point $A = (1,2)$, has length 3 cm in the direction N 60° E.

   b. $\overrightarrow{OR}$ where O is the origin and it is 5 cm to the North.

   c. $\overrightarrow{OS}$ whose initial is at the origin and has a length 4 cm in the direction S 30° E .

8.3 Vector Operations

In section 8.1 and section 8.2, you have studied the definition of a vector, different types of vectors and its representation. In this section, you will learn about operations on vectors.

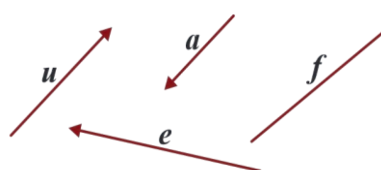
8.3.1 Addition of vectors

Activity 8.2

1. Consider the following vectors

   a. Is it possible to add vectors like $\boldsymbol{a}$ and $\boldsymbol{e}$?

   b. Can you subtract $\boldsymbol{e}$ from $\boldsymbol{f}$?



2. Suppose a student start walking from her home to school which is located at a distance of 800 m to the East. Then she went to the market 400 m far from the school to the North. Finally, she returned to her home.

   a. Discuss the student's journey using vector representation.

   b. What is the total distance covered by the student?

Your discussion in the activity leads you to the importance of addition and subtraction of vectors. There are two laws of addition of vectors:

i) Triangle law of addition of vectors

Consider two vectors as given in figure 8.15.

Now we need to add these two vectors $\boldsymbol{u}$ and $\boldsymbol{v}$. To get their sum, $\boldsymbol{v}$ is translated, keeping its direction and length unchanged, until its initial coincides with the terminal point of $\boldsymbol{u}$.

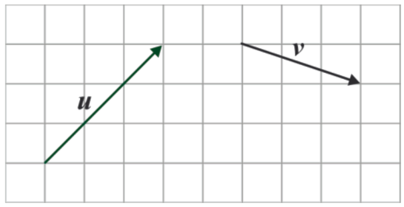


Figure 8.15

Then, the sum of $\boldsymbol{u}+\boldsymbol{v}$ is defined by the vector represented by the third side of the completed triangle, that is $\boldsymbol{w}$ as shown in figure 8.16.

Note that, from figure 8.16, we can write $\boldsymbol{w}=\boldsymbol{u}+\boldsymbol{v}$ since going along vector $\boldsymbol{u}$ and then along vector $\boldsymbol{v}$ is equivalent to going along vector $\boldsymbol{w}$.

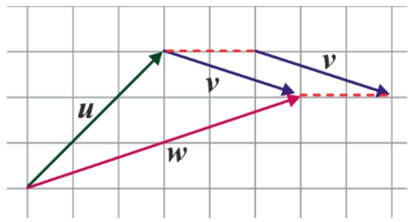
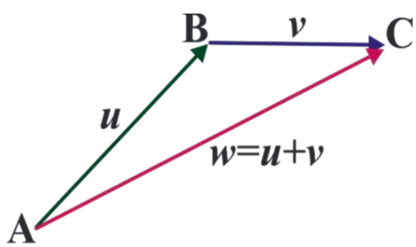


Figure 8.16

Definition 8.3 Triangle law of vector addition

Consider two vectors $\overrightarrow{AB}=\vec{u}$ and $\overrightarrow{BC}=\vec{v}$ in a coordinate system. The sum of $\overrightarrow{AB}+\overrightarrow{BC}=\vec{u}+\vec{v}$ is a directed line segment connecting A to C say $\vec{w}=\overrightarrow{AC}$ such that $\overrightarrow{AB}+\overrightarrow{BC}=\overrightarrow{AC}$ or $\vec{u}+\vec{v}=\vec{w}$. Such a vector $\overrightarrow{AC}$ is called a ***resultant vector***.



Addition of vectors can be done graphically, and addition of vector can be done using components. Now, we will find the sum of vectors by graphical method. The second method will be presented after section 8.4.

Example

By using figure 8.17 , determine each of the following.

a. $\overrightarrow{FH} + \overrightarrow{HG}$

b. Compare $\overrightarrow{EF} + \overrightarrow{FG}$ and $\overrightarrow{GH} + \overrightarrow{HE}$

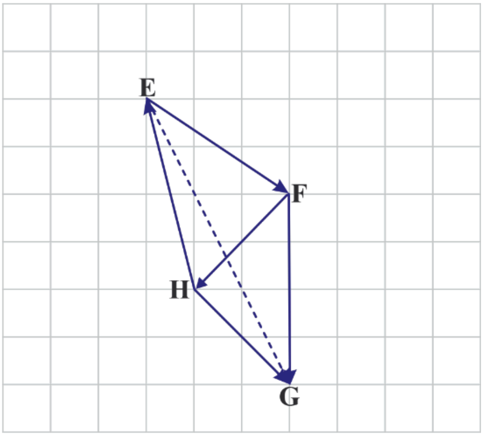


Figure 8.17

Solution:

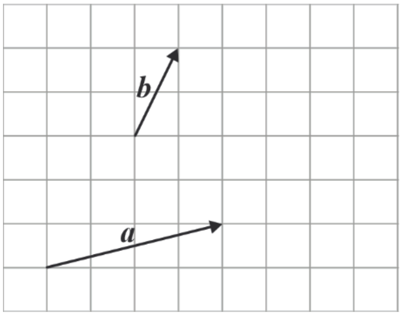
a. $\overrightarrow{FH} + \overrightarrow{HG} = \overrightarrow{FG}$

b. $\overrightarrow{EF} + \overrightarrow{FG} = \overrightarrow{EG}$ and $\overrightarrow{GH} + \overrightarrow{HE} = \overrightarrow{GE}$, the two vectors are parallel and opposite.

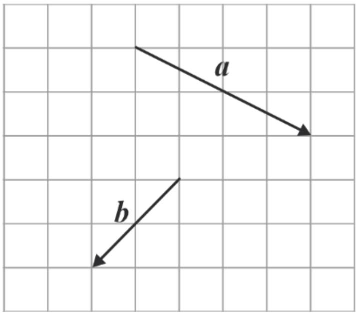
Exercise 8.5

1. Draw the vectors $\boldsymbol{a} + \boldsymbol{b}$ using the triangle law.

(a)



(b)

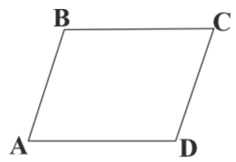


ii) Parallelogram law of vector addition:

For two or more free vectors with different initial points, you have learned how to determine the resultant vector (sum of two vectors) by joining the initial point of the first vector to the terminal point of the second. But, there are co-initial vectors where you may be asked to determine their sum.

Activity 8.3

1. What is a parallelogram?
2. What are the properties of corresponding sides of parallelogram?



Let us consider two vectors with the same initial point (A) and make a non-zero angle between them as shown in figure 8.18.

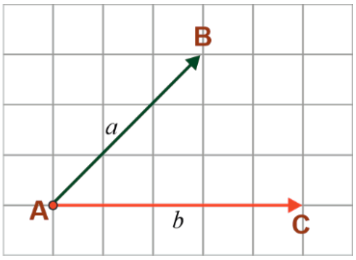


Figure 8.18

Can you construct a parallelogram using these two intersecting vectors? Recall, the definition of equal vectors. Construct a broken line passing through B and parallel to vector $\vec{b}$. Similarly construct a broken line through C and parallel to vector $\vec{a}$. The two broken lines intersect at a point, say $D,$ as shown in figure 8.19.

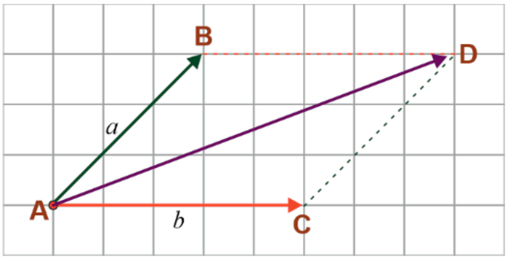


Figure 8.19

Did you observe a parallelogram and two pairs of equal vectors in this figure? $\overrightarrow{AB}$ & $\overrightarrow{CD}$, $\overrightarrow{AC}$ & $\overrightarrow{BD}$ are equal vectors. By triangle law of vector addition, the resultant vector will be $\overrightarrow{AB}+\overrightarrow{BD}=\overrightarrow{AC}+\overrightarrow{CD}=\overrightarrow{AD}$. That is, $\boldsymbol{a}+\boldsymbol{b}=\overrightarrow{AD}$. This method of determining the sum of vectors is called ***parallelogram law of vector addition***.

Note

If two vectors have different initial points, you need to bring the two initial points placed at a point by construction before applying parallelogram method.

Subtraction of vectors

How do you subtract a vector from another vector?

Subtraction of vectors can be treated as addition of vector and a negative vector. Consider two vectors which are to be subtracted as shown in figure 8.20. To determine the resultant vector (the difference of the two vectors) , we follow the following procedures.

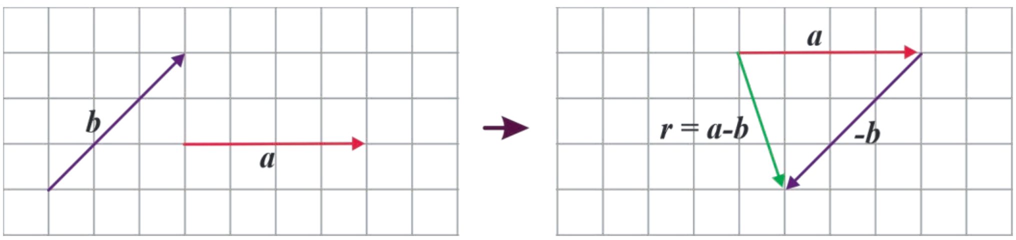


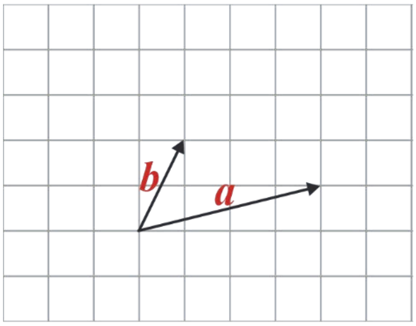
Figure 8.20

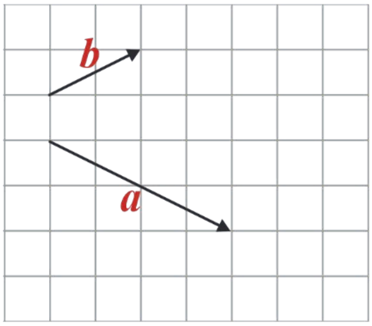
- The first vector is drown with proper scale and in a given direction.
- Then, from the terminal point of the first vector ($\vec{a}$), a vector is drown with the same scale and in the opposite direction of the second vector (-$\vec{b}$).
- Then, the vector joining the initial point of the first vector and the terminal pint of the second vector represent the resultant vector ($\vec{r}$) with direction and magnitude.

Exercise 8.6

1. Draw the vectors $\boldsymbol{a}+\boldsymbol{b}$ and $\boldsymbol{a}-\boldsymbol{b}$ using the parallelogram law or the triangle law of vector addition.

(a) (b)





Addition and subtraction of column vectors

For two vectors in component form $\begin{pmatrix} p \\ q \end{pmatrix}$ and $\begin{pmatrix} r \\ s \end{pmatrix}$, we have

$$\begin{pmatrix} p \\ q \end{pmatrix} + \begin{pmatrix} r \\ s \end{pmatrix} = \begin{pmatrix} p+r \\ q+s \end{pmatrix} \text{ and } \begin{pmatrix} p \\ q \end{pmatrix} - \begin{pmatrix} r \\ s \end{pmatrix} = \begin{pmatrix} p-r \\ q-s \end{pmatrix}.$$

Example 1

Consider column vectors $\boldsymbol{a} = \begin{pmatrix} -4 \\ -3 \end{pmatrix}$ and $\boldsymbol{b} = \begin{pmatrix} 5 \\ 0 \end{pmatrix}$ .Then find

a) $\boldsymbol{a} + \boldsymbol{b}$ b) $\boldsymbol{a} - \boldsymbol{b}$

Solution:

a. $\boldsymbol{a} + \boldsymbol{b} = \begin{pmatrix} -4 \\ -3 \end{pmatrix} + \begin{pmatrix} 5 \\ 0 \end{pmatrix} = \begin{pmatrix} -4+5 \\ -3+0 \end{pmatrix} = \begin{pmatrix} 1 \\ -3 \end{pmatrix}$. This can also be observed from the following figure 8.21.

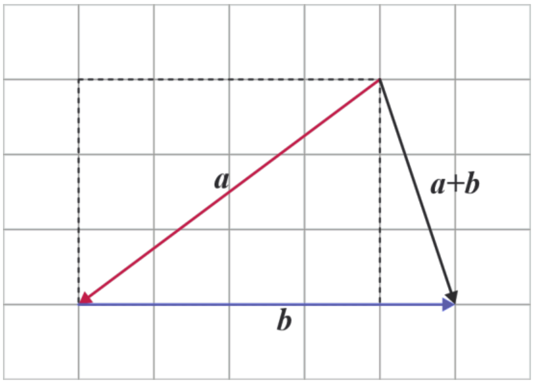


Figure 8.21

b. $\boldsymbol{a}-\boldsymbol{b}=\binom{-4}{-3}-\binom{5}{0}=\binom{-4-5}{-3-0}=\binom{-9}{-3}$ as shown in figure 8.22.

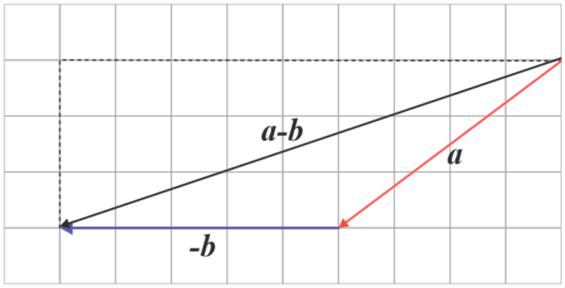


Figure 8.22

Example 2

Consider the quadrilateral as shown in the following figure 8. 23. Given that $\overrightarrow{AB}=\boldsymbol{a}, \overrightarrow{DC}=\boldsymbol{b}$ and $\overrightarrow{DB}=\boldsymbol{c}$, then find the following in terms of $\boldsymbol{a}, \boldsymbol{b}$ and $\boldsymbol{c}$.

i) $\overrightarrow{AC}$ ii) $\overrightarrow{AB}$ iii) $\overrightarrow{BC}$

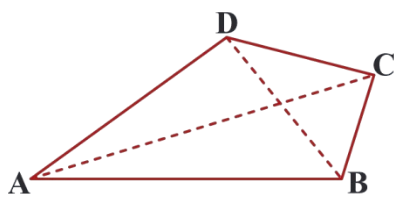


Figure 8.23

Solution:

i. From the figure $\overrightarrow{AC}=\boldsymbol{a}+\boldsymbol{b}$

ii. $\overrightarrow{AB}=\boldsymbol{a}+\boldsymbol{c}$

iii. $\overrightarrow{BC}=\boldsymbol{b}-\boldsymbol{c}$

Exercise 8.7

1. For the vectors $\boldsymbol{a}=\binom{1}{3}$, $\boldsymbol{b}=\binom{2}{-1}$ and $\boldsymbol{c}=\binom{-4}{5}$, find

a) $\boldsymbol{a}+\boldsymbol{b}$ b) $\boldsymbol{a}-\boldsymbol{b}$

c) $\boldsymbol{a}+\boldsymbol{c}$ d) $\boldsymbol{b}-\boldsymbol{c}$

e) $\boldsymbol{a}+\boldsymbol{b}+\boldsymbol{c}$

2. $ABCD$ is a rhombus , $\triangle DBC$, $\triangle ABD$ and $\triangle BCE$ are equilateral triangles.

Suppose $\overrightarrow{AD}=\boldsymbol{a}$, $\overrightarrow{AB}=\boldsymbol{b}$ and $\overrightarrow{BD}=\boldsymbol{c}$.

Find each in terms of $\boldsymbol{a}$, $\boldsymbol{b}$ and $\boldsymbol{c}$

i) $\overrightarrow{AC}$ ii) $\overrightarrow{ED}$

iii) $\overrightarrow{BC}$ iv) $\overrightarrow{CE}$

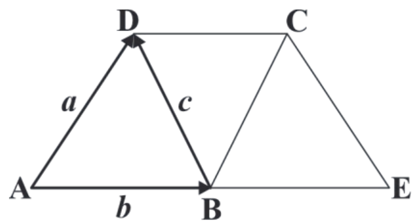


Figure 8.24

3. Using the following figure 8.25,
   a. Write $\overrightarrow{AC}$ as a sum of two vector.
   b. Describe $\overrightarrow{BD}$ as a difference of two vectors.
   c. What is the sum of $\overrightarrow{AB}$, $\overrightarrow{BC}$ and $\overrightarrow{CD}$?

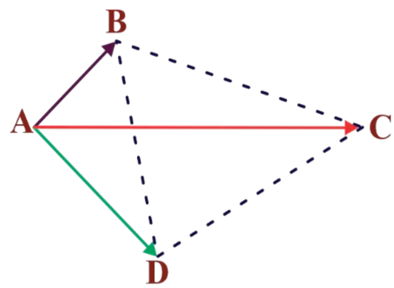


Figure 8.25

4. Given $ABDC$ is a parallelogram
   a. By construction show that $\overrightarrow{AB} - \overrightarrow{AC} = \overrightarrow{CB}$.
   b. Construct the diagonals of the parallelogram and show that these vectors are the sum and difference of the adjacent side vectors of the parallelogram

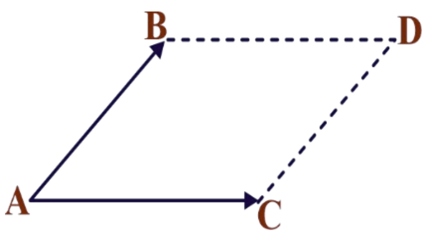


Figure 8.26

8.3.2 Multiplication of a vector by a scalar

In the previous section, you have studied how to get the sum and difference of vectors geometrically. In this section, you will learn scalar multiplication of vectors

Activity 8.4

Using figure 8.27 below, express vectors $\boldsymbol{v}, \boldsymbol{m}, \boldsymbol{n}$ and $\boldsymbol{p}$ in terms of a vector $\boldsymbol{u}$.

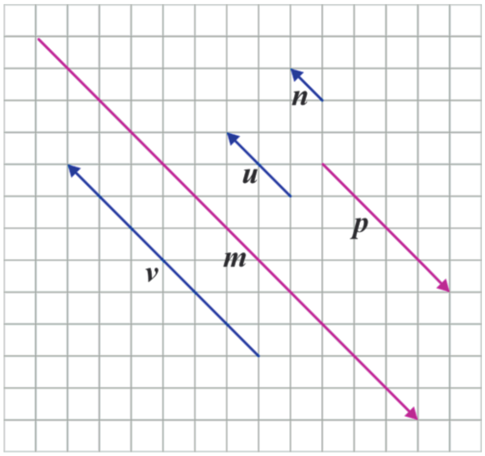


Figure 8.27

What do you conclude from your activity? All the given vectors are parallel and you can express the length of each vector interms of the length of $\boldsymbol{u}$. For example, $\boldsymbol{v}$ is in the same direction as $\boldsymbol{u}$ and its magntude is 3 times that of length of $\boldsymbol{u}$. Hence, it is possible to write $\boldsymbol{v} = \boldsymbol{3u}$.

Definition 8.4

Let $\overrightarrow{AB}$ be any given vector and k be any real number. The scalar multiple vector $k\overrightarrow{AB}$ is the vector whose magnitude (length) is k times the magnitude of $\overrightarrow{AB}$ and

i. the direction of $k\overrightarrow{AB}$ is the same as the direction of $\overrightarrow{AB}$, if $k > 0$,

ii. the direction of $k\overrightarrow{AB}$ is the opposite of $\overrightarrow{AB}$, if $k < 0$.

Example

Given a vector $\overrightarrow{AB}$ as shown in Figure 8.28 . Draw a vector $2\overrightarrow{AB}$, $\frac{1}{2}\overrightarrow{AB}$, $-2\overrightarrow{AB}$ and $\frac{-3}{2}\overrightarrow{AB}$.

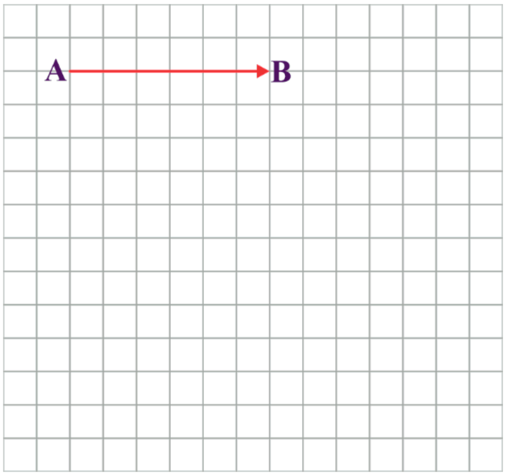


Figure 8.28

Solution:

Using definition 8.4, we can draw the required vectors as follows.

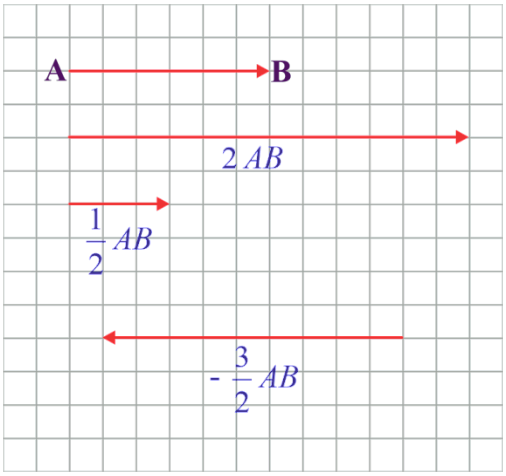


Figure 8.29

Note

Two vectors are said to be parallel if one can be written as a scalar multiple of the other. If $\vec{b}$ is parallel to $\vec{a}$, then it can be expressed as $\vec{b} = k\vec{a}$.

Exercise 8.8

1. Given $\overrightarrow{AB}$ as shown in figure 8.30, draw vector

   a. $2\overrightarrow{AB}$ b. $\frac{1}{2}\overrightarrow{AB}$ c. $-3\overrightarrow{AB}$

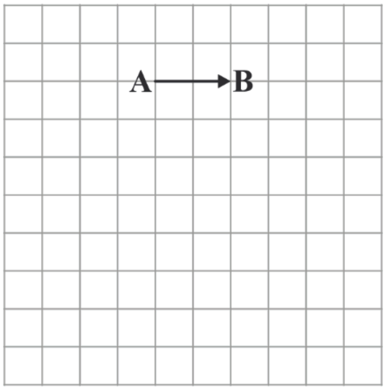


Figure 8.30

Example 2

Consider column vector $\boldsymbol{a} = \begin{pmatrix} 2 \\ 4 \end{pmatrix}$ .Then find $-\boldsymbol{a}$, $2\boldsymbol{a}$ and $\frac{1}{2}\boldsymbol{a}$

Solution:

For the vector in component form $\begin{pmatrix} a \\ b \end{pmatrix}$,

$k\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} ka \\ kb \end{pmatrix}$ where k is a scalar.

So that for $\boldsymbol{a} = \begin{pmatrix} 2 \\ 4 \end{pmatrix}$, $-\boldsymbol{a} = -1\begin{pmatrix} 2 \\ 4 \end{pmatrix} = \begin{pmatrix} -2 \\ -4 \end{pmatrix}$,

$2\boldsymbol{a} = 2\begin{pmatrix} 2 \\ 4 \end{pmatrix} = \begin{pmatrix} 4 \\ 8 \end{pmatrix}$ and $\frac{1}{2}\boldsymbol{a} = \frac{1}{2}\begin{pmatrix} 2 \\ 4 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$. This is illustrated in the following Figure 8.31. From the figure, you can observe that

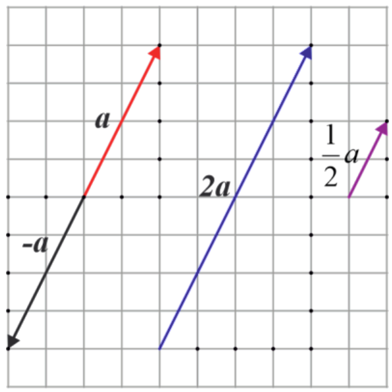


Figure 8.31

$\boldsymbol{a}, -\boldsymbol{a}$, $2\boldsymbol{a}$ and $\frac{1}{2}\boldsymbol{a}$ are parallel to each other.

Exercise 8.9

1. Given vectors $\boldsymbol{a} = \begin{pmatrix} -1 \\ 4 \end{pmatrix}$ and $\boldsymbol{b} = \begin{pmatrix} 3 \\ 6 \end{pmatrix}$. Then find

   a. $4\boldsymbol{a}$ b. $-3\boldsymbol{a}$ c. $\frac{1}{3}\boldsymbol{b}$ d. $-\frac{1}{3}\boldsymbol{b}$

2. Write true if the statement is correct and false otherwise.

   a. A vector is a scalar multiple of its opposite vector.

   b. For a given vector $\vec{u}$ and scalar k, $k\vec{u}$ has larger magnitude than $\vec{u}$ for $k > 0$.

c. If the magnitude of two vectors is the same, then they have the same direction.

d. A vector is parallel to itself.

3. From the following figure 8.32, describe vectors $\overrightarrow{AC}, \overrightarrow{AD}, \overrightarrow{AF}$ and $\overrightarrow{AG}$, each vector as a scalar multiple of $\overrightarrow{AB}$.

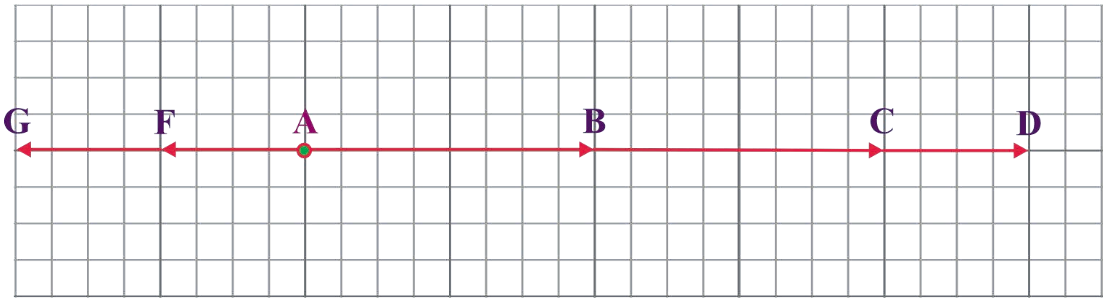


Figure 8.32

8.4 Position Vector

In the previous 3 sections you practiced how to represent vectors geometrically and vector operations. In this section you will discuss representing a vector with a fixed initial point and the use of component of a vector to determine its magnitude and position.

Activity 8.5

1. On the coordinate system, there are different vectors as shown in figure 8.33.
   a. What can you say about the two vectors q and p?
   b. What can you say about vectors v and $\vec{w}$?
   c. Is it possible to bring (move) the initial of r to the initial of v ?
   d. What makes vectors v and $\vec{w}$ differ from other vectors?

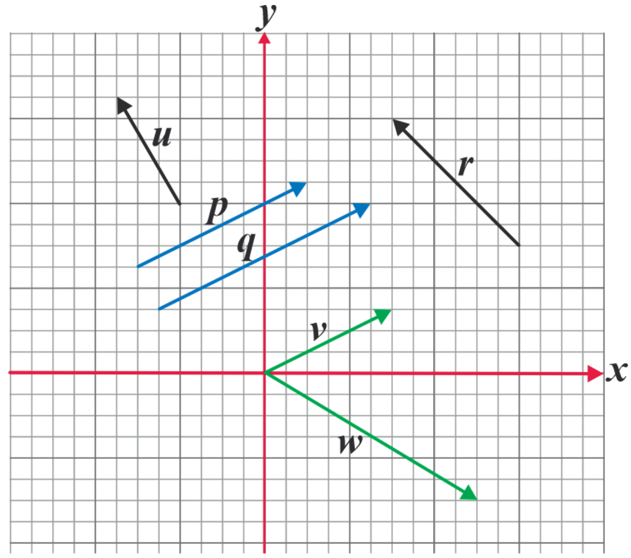


Figure 8.33

2. Suppose $\overrightarrow{AB}$ be a vector with initial point at $A = (1,2)$ and the terminal point $B = (3,5)$. If the initial point moves to the point $O = (0,0)$.
   a. What will be the location of the terminal point (say, point C)?
   b. What is the relationship between $\overrightarrow{AB}$ and $\overrightarrow{OC}$?

From your discussion in the activity, you have seen how to move the initial point of a vector to the origin (a point on a plane with order pair $O = (0,0)$).

Now, let us define what a position vectors mean.

Definition 8.5

Vectors that start at the origin (O) are called position vectors.

The position vectors are also said to be vectors which are written in the standard form. These are used to determine the position of a point with reference to the origin.

How can we determine the position vector of any vector with initial point $A = (x_1, y_1)$ and terminal point $B = (x_2, y_2)$?

To find the position vector corresponding to vector $\overrightarrow{AB}$, subtract the corresponding components of A from B as $(x_2 - x_1, y_2 - y_1)$. Let this point be $P = (x, y)$. Now, we have two equal vectors $\overrightarrow{AB}$ *and* $\overrightarrow{OP}$ with the same component, where $\overrightarrow{OP}$ is the position vector of $\overrightarrow{AB}$.

Let $(x_2 - x_1, y_2 - y_1) = (x, y)$, so that $\overrightarrow{OP}$ has components x and y as shown in the following figure 8.34.

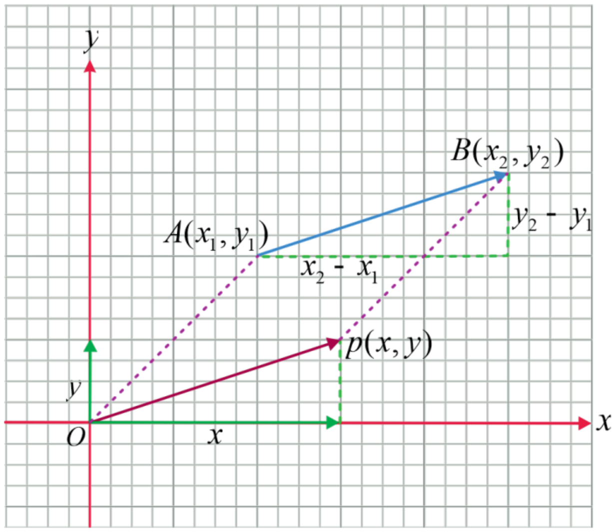


Figure 8.34

Notation

A position vector $\overrightarrow{OP}$ can be represented by $\overrightarrow{OP} = \vec{u} = \binom{x}{y}$, $\vec{u}$ is expressed as a sum of two vectors $\vec{x}$ and $\vec{y}$ as $\vec{u} = \vec{x} + \vec{y}$ from triangle or parallelogram law where $\vec{x} = x\vec{\imath}$ *and* $\vec{y} = y\vec{\jmath}$. The vectors $\vec{\imath} = \binom{1}{0}$ *and* $\vec{\jmath} = \binom{0}{1}$ are vectors with 1 unit long in the x and y directions, respectively. Hence, the given vector $\vec{u}$ can be written as $\vec{u} = x\vec{\imath} + y\vec{\jmath}$, in this case x and y are called components of $\vec{u}$.

The magnitude of the position vector $|\vec{u}| = \sqrt{x^2 + y^2}$.

Example

Consider the position vector $\vec{u} = \binom{3}{4}$.

a. Draw this vector on the coordinate plane.

b. Find $|\vec{u}|$.

c. Determine the direction of $\vec{u}$.

Solution:

a. Since this is a position vector, it starts from the origin to its terminal point (3,4). It can be written as $\vec{u} = 3\vec{\imath} + 4\vec{\jmath}$. This is to mean, move 3 units to the right and then 4 units up as shown in figure 8.34.

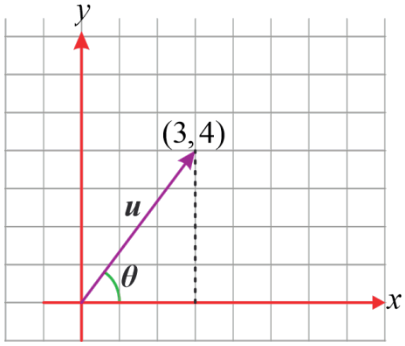


Figure 8.35

b. The magnitude of the vector is

$\sqrt{3^2 + 4^2} = \sqrt{25} = 5.$

c. By definition $\tan\theta = \frac{4}{3}$. From trigonometric table, we can read that $\theta \approx 53°$.

Exercise 8.10

1. Draw the following vector on a coordinate plane. Then, find the magnitude.

a. $\vec{u} = \binom{2}{3}$ b. $\vec{v} = \binom{-1}{2}$ c. $\vec{w} = \binom{4}{-3}$ d. $\vec{n} = \binom{-1}{-5}$

Example

Determine the position vector formed by the given initial and terminal point.

a. Initial point (1,3), terminal pint (2,2).

b. Initial point (−1, −2), terminal point (3,1).

Solution:

a. The position vector formed by the given initial point (1,3) and terminal point (1,2) is $(2-1, 2-3) = (1, -1)$. This position vector can be written as $\vec{u} = 1\vec{\imath} - 1\vec{\jmath}$ which

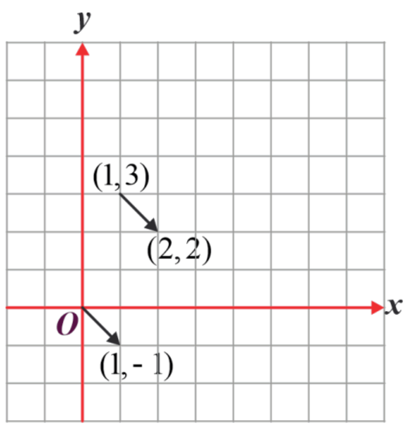


Figure 8.36 (a)

could be represented as $\vec{u} = \binom{1}{-1}$. (See figure 8.36 (a)).

b. The position vector formed by the given initial point $(-1,-2)$ and terminal point $(3,1)$ will be $(3+1, 1+2) = (4,3)$. This position vector can be written as $\vec{u} = 4\vec{\imath} + 3\vec{\jmath}$ which can be written as $\vec{u} = \binom{4}{3}$. (see figure 8.36 (b)).

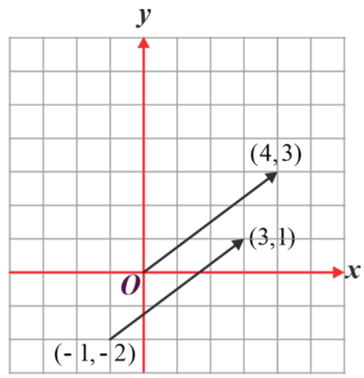


Figure 8.36 (b)

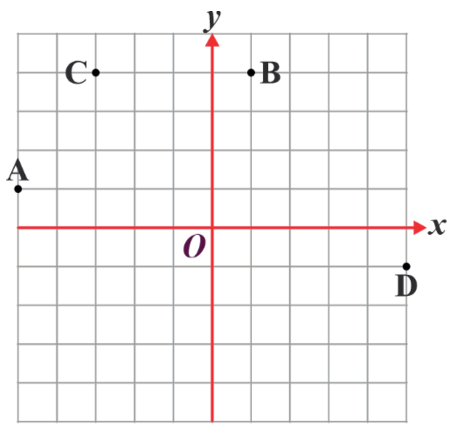
Exercise 8.11

Given points $A = (-5,1)$, $B = (1,4)$, $C = (-3,4)$ *and* $D = (5,-1)$ be points on the coordinate system.

1. i. Find the position vector whose initial point is A and terminal point B.

   ii. Find the position vector whose initial point is C and terminal point is D.

2. Determine the magnitude and direction of $\overrightarrow{BC}$ and $\overrightarrow{AB}$ using the components of the position vectors corresponding to the given vectors.



8.5 Applications of Vectors in Two Dimensions

Until now, you have studied vector operation and expressed a vector as a position vector and determining its components. In this section, we will see some application of vectors in two dimensions.

Example 1

Suppose a boat start moving on the sea 8 km to the South and then 8 km to the West. What is the displacement of the boat from its starting position to its last destination?

Solution:

Let the starting point of the boat be A and the first destination be B, and its last destination be C as shown in figure 8.37.

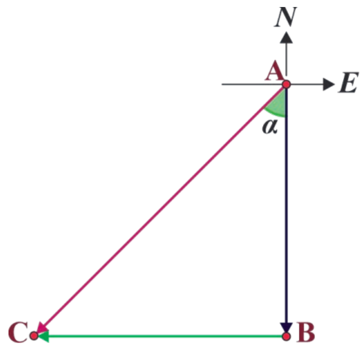


Figure 8.37

The magnitude $AC = \sqrt{(AB)^2 + (BC)^2}$

$= \sqrt{64 + 64} = 8\sqrt{2}$

and $\tan\alpha = \frac{\text{opposite}}{\text{adjacent}} = \frac{8}{8} = 1.$

So that, $\alpha = 45°$. Hence, the displacement of $\overrightarrow{AC}$ is $8\sqrt{2}$ km in the S 45° W direction.

Example 2

$ABCD$ is a rectangle, point E is midpoint of AB, point F is midpoint of BC, point G is midpoint of CD, point H is midpoint of AD. $\overrightarrow{AE} = \boldsymbol{a}$ and $\overrightarrow{AH} = \boldsymbol{b}$ as shown in figure 8.38 below.

a. Express each of the following interms of $\boldsymbol{a}$ and $\boldsymbol{b}$

i) $\overrightarrow{AF}$ ii) $\overrightarrow{BD}$

iii) $\overrightarrow{EC}$

b. Show that $\overrightarrow{BD}$ is parallel to $\overrightarrow{EH}$

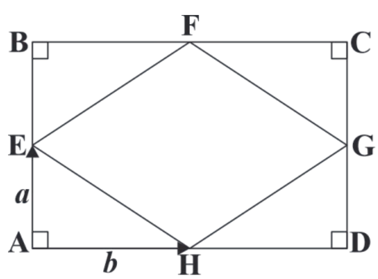


Figure 8.38

Solution:

a. i) $\overrightarrow{AF} = \overrightarrow{AB} + \overrightarrow{BF} = 2\boldsymbol{a} + \boldsymbol{b}$

ii) $\overrightarrow{AB} + \overrightarrow{BD} = \overrightarrow{AD}$ so that

$\overrightarrow{BD} = \overrightarrow{AD} - \overrightarrow{AB} = 2\boldsymbol{b} - 2\boldsymbol{a}$

iii) $\overrightarrow{EC} = \overrightarrow{EB} + \overrightarrow{BC} = \boldsymbol{a} - 2\boldsymbol{b}$

b. From above part a. (ii) , $\overrightarrow{BD} = 2\boldsymbol{b} - 2\boldsymbol{a}$. $\overrightarrow{AB} + \overrightarrow{EH} = \overrightarrow{AH}$, so that

$\overrightarrow{EH} = \overrightarrow{AH} - \overrightarrow{AB} = \boldsymbol{b} - \boldsymbol{a}$. $\overrightarrow{BD} = 2(\boldsymbol{b} - \boldsymbol{a}) = 2\overrightarrow{EH}$.

Therefore $\overrightarrow{BD}$ is parallel to $\overrightarrow{EH}$.

Example 3

Two forces, one 10 N and the other $10\sqrt{3}$ N, are acting on the body. Given that the two forces are acting perpendicularly to each other, find the magnitude of the third force which would just counter the two forces.

Solution:

Let us say, the first force be $F_1 = 10$ N and the second be $F_2 = 10\sqrt{3}$ N as shown in the following figure 8.39.

Using parallelogram law, the resultant force $\vec{R} = \vec{F_1} + \vec{F_2}$. Since the two forces are acting perpendicularly, the magnitude of the resultant force is determined by $|\vec{R}| = \sqrt{|\vec{F_1}|^2 + |\vec{F_2}|^2}$.

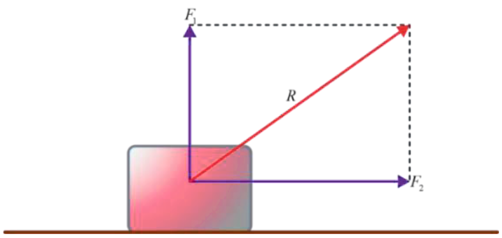


Figure 8.39

That is $|\vec{R}| = \sqrt{10^2 + \left(10\sqrt{3}\right)^2}$

$= \sqrt{400} = 20.$

Hence, the magnitude of the resultant force is 20 N.

Exercise 8.12

1. Express each of the following in terms of $\boldsymbol{a}, \boldsymbol{b}$ and $\boldsymbol{c}$, using the following figure 8.40

   a. $\overrightarrow{AC}$ b. $\overrightarrow{AD}$

   c. $\overrightarrow{AE} + \overrightarrow{ED}$ d. $\overrightarrow{BD}$

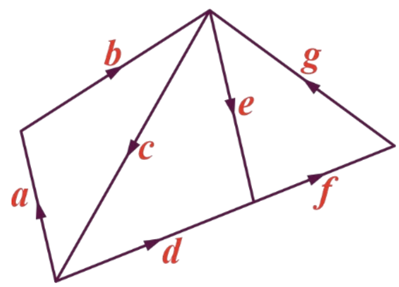


Figure 8.40

2. A wagon is being pulled by a rope that makes a $60°$ with the ground. A person is pulling with a force of $100\,N$ along the rope. Determine the horizontal and vertical components of the vector.
3. A swimmer heads directly to the north across a river swimming at 1.6 m/s relative to still water which moves to the east. She arrives at a point 40 m downstream from the point directly across the river, which is 80 m wide. Determine the displacement of the swimmer.

Summary

1. Scalars are quantities that are fully described by a magnitude (or numerical value) alone.
2. Vectors are quantities that are fully described by both a magnitude and a direction.
3. A vector is represented geometrically by a directed line segment (a line segment with direction) denoted by an arrow$\overrightarrow{AB}$. In this case, the point A is called the *initial* and the point B is called the *terminal.* A vector also represented by using letters with an arrow bar over it such as $\vec{u}, \vec{v}, \vec{a}$ etc.
4. The magnitude of a given vector $\overrightarrow{AB}$ or $\overrightarrow{u}$ is the length of the line segment from its initial point A to its terminal point. It is denoted by absolute value sign as $|\overrightarrow{AB}|$or $|\vec{u}|$.
5. On a plane, the direction of a vector is given by the angle the vector makes with a reference direction, often an angle with the horizontal or with the vertical.
6. Vectors that have the same or opposite direction are called parallel vectors.
7. Two vectors are said to be equal if they have the same magnitude and the same direction.
8. For any two vectors $\overrightarrow{AB}$ and $\overrightarrow{BC}$, $\overrightarrow{AB} + \overrightarrow{BC} = \overrightarrow{AC}$ (the Triangle Law)
9. A vector that has no magnitude and direction is called a zero vector or null vector
10. The diagonal of a parallelogram is the sum of the side vectors. This is called Parallelogram Law.
11. Any vector can be enlarged or shortened, by multiplying the vector by a scalar. If $\vec{u}$ is a vector and k be a scalar
    a. if $|k| > 1$, it enlarges the vector
    b. if $0 < |k| < 1$, it shortens the vector

c. if $k > 0$, a scalar multiple of $\vec{u}$ $(k\vec{u})$ is in the same direction to $\vec{u}$.

d. if $k < 0$, a scalar multiple of $\vec{u}$ $(k\vec{u})$ is in the opposite direction to $\vec{u}$.

12. A vector that starts from the origin (O) is called a *position vector*. The position vector $\vec{u}$ can be written as $\vec{u} = x\vec{\imath} + y\vec{\jmath} = \begin{pmatrix} x \\ y \end{pmatrix}$ where $\vec{\imath} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ *and* $\vec{\jmath} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ are vectors whose magnitude is 1 in the direction of x *and* y axis respectively, x *and* y are called components of the vector $\vec{u}$. The magnitude of $\vec{u}$ is determined by $|\vec{u}| = \sqrt{x^2 + y^2}$.

13. Let the initial and terminal points of a vector are (x_1, y_1) and (x_2, y_2) then its position vector can be calculated as $P = (x_2 - x_1, y_2 - y_1) = \begin{pmatrix} x_2 - x_1 \\ y_2 - y_1 \end{pmatrix}$.

Review Exercise

1. Define scalar and vector quantities.
2. List out some scalar and vector quantities.
3. Find the magnitude of each of the following vectors where each grid is of square unit.

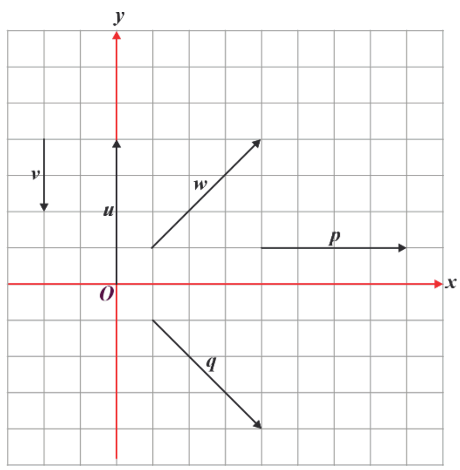


Figure 8.41

4. Consider the points $A = (2,1), B = (10,6), C = (13,4)$ and $D = (16,-2)$. Determine the components form of vector $\overrightarrow{AD}, \overrightarrow{BC}, \overrightarrow{CD}$ *and* $\overrightarrow{AB}$.

5. Simplify (using Figure 8.42)

a. $\boldsymbol{c}+\boldsymbol{f}$ **c.** $\boldsymbol{g}-\boldsymbol{f}$

b. $\boldsymbol{c}+\boldsymbol{d}+\boldsymbol{e}$ **d.** $\boldsymbol{d}+\boldsymbol{e}$

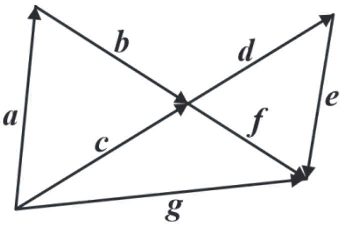


Figure 8.42

6. Simplify

a. $\overrightarrow{AB}+\overrightarrow{BD}$ **b.** $\overrightarrow{PQ}+\overrightarrow{QM}+\overrightarrow{MN}$ **c.** $\overrightarrow{LM}+\overrightarrow{MN}+\overrightarrow{NS}+\overrightarrow{SK}$

7. Use the figure 8.43 to name the directed line segment

i. $\boldsymbol{a}+\boldsymbol{b}$

ii. $\boldsymbol{a}+\boldsymbol{b}+\boldsymbol{e}$

iii. $\boldsymbol{g}+\boldsymbol{e}$

iv. $\boldsymbol{c}+\boldsymbol{d}$

v. $\boldsymbol{e}+\boldsymbol{f}$

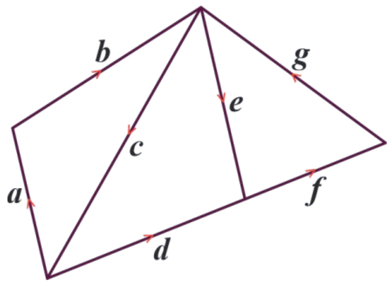


Figure 8.43

8. Suppose $A=(0,1), B=(0,3)$ be points on the coordinate system and $\vec{u}$ be a position vector whose terminal point is $(6,8)$. Compare the magnitude of $\overrightarrow{AB}$ and $\vec{u}$.

9. Sketch a vector of length 3cm in the direction of

a. South **b.** South 30° West **c.** North 45° East

10. Let A and B be two vectors given by $A=3i-2j$ and $B=5i-\frac{1}{2}j$ then the vector $\frac{1}{2}A-\frac{2}{3}B=$ __________.

A. $-\frac{11}{6}i-\frac{4}{3}j$ **B.** $8i-\frac{3}{2}j$ **C.** $-\frac{11}{6}i-\frac{2}{3}j$ **D.** $-\frac{29}{6}i-\frac{4}{3}j$

11. Seven vectors are drawn as shown in the figure below (figure 8.44), where $PQRS$ is a parallelogram and $\overrightarrow{PT}=-\overrightarrow{QS}$. Which one of the following is true about these vectors?

A. $\overrightarrow{PR} - \overrightarrow{QR} = \overrightarrow{QP}$

B. $\overrightarrow{PQ} - \overrightarrow{PS} = \overrightarrow{PT}$

C. $\overrightarrow{PS} - \overrightarrow{PT} = \overrightarrow{RS}$

D. $\overrightarrow{PS} - \overrightarrow{SR} = \overrightarrow{PR}$

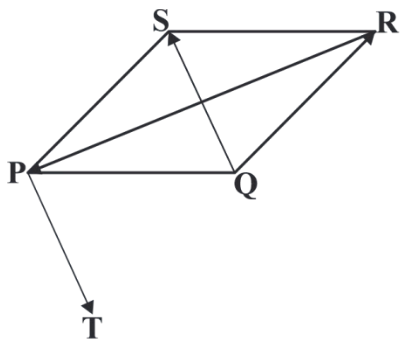


Figure 8.44

12. If the position vector V is given by $V = 5i + j$, then which one of the following is equal to V?

A. $\overrightarrow{EF}$ where $E(-1,-2)$ and $F(0,3)$

B. $\overrightarrow{GH}$ where $G(3,-1)$ and $H(2,2)$

C. $\overrightarrow{PQ}$ where $P(-3,-4)$ and $Q(2,-1)$

D. $\overrightarrow{RS}$ where $R(5,6)$ and $S(10,7)$

13. The vector $\vec{v}$ has initial point $P = (1,0)$ and terminal point Q that is on the $y - axis$ and above the initial point. Find coordinates of terminal point Q such that the magnitude of the vector $\vec{v}$ is $\sqrt{10}$.

14. Let $\vec{a}$ be a standard-position vector with terminal point$(-2,-4)$. Let $\vec{b}$ be a vector with initial point $(1,2)$ and terminal point $(-1,4)$. Find the magnitude of $-3\vec{a} + \vec{b} - 4\vec{i} + \vec{j}$.

15. A merchant start to move in a city. He drives to the bookstore which is 17 km to the north. Then he moves to the music shop 6 km to east and finally he drives 5km to south. Find the displacement of the merchant.

16. A vector $\vec{u}$ is directed along 30° west of north and another vector $\vec{v}$ along 75° east of south. In which of the direction of the resultant of the two vectors could not exist? Justify your answer.

A. North　　B. North-East　　C. East　　D. South

17. A plane flies 100 km, N30°W and after a brief stopover flies150 km, N60°E. Determine the plane's displacement.

UNIT 9

STATISTICS AND PROBABILITY

Unit Outcomes

By the end of this unit, you will be able to

- **Collect and represent simple statistical data using different methods such as histograms.**
- **Use facts and basic principles of probability.**
- **Develop the concept of probability via experimentation and hypothetical events.**

Unit Contents

KEY Words

- outcomes
- presentation
- event
- population
- frequency
- histogram
- interpretation
- analysis
- range
- median
- sample
- raw data
- mode
- Sample space
- equally likely
- tabulation
- variance
- probability
- average
- Measure of central tendency
- measure of dispersion
- classification of mode
- descriptive statistics
- secondary data
- Standard deviation
- statistical data
- frequency distribution
- primary data
- variable
- Arithmetic mean

INTRODUCTION

Activity 9.1

a. Classify the students in your class by their age. What is the average age of the class?

b. Classify the length of time each student in your class takes to get to school. Identify the length of time that is most frequent.

c. Classify time spent to help their parents by each student in your class.

Everybody collects, interprets and uses information; much of it is in numerical or statistical forms in day-to-day life. It is a common practice that people receive large quantities of information everyday through conversations, televisions, computers, the radios, newspapers, posters, notices and instructions. It is just because there is so much information available that people need to be able to absorb, select and reject it. In everyday life, in business and industry, certain statistical information is necessary and it is independent to know where to find it how to collect it. As a consequence, everybody has to compare prices and quality before making any decision about what goods to buy. As employees of any firm, people want to compare their salaries and working conditions, promotion opportunities and so on. In time the firms on their part want to control costs and expand their profits. One of the main functions of statistics is to provide information which will help on making decisions. Statistics provides the type of information by providing a description of the present, a profile of the past and an estimate of the future.

The following are some of the objectives of collecting statistical information.

1. To describe the methods of collecting primary statistical information.
2. To consider the status involved in carrying out a survey.
3. To analyze the observation and interpreting it.
4. To define and describe sampling.
5. To analyze the basis of sampling.
6. To describe a variety of sampling methods.

Statistical investigation is a comprehensive activity and requires systematic collection of data about some group of people or objects, describing and organizing the data, analyzing the data with the help of different statistical methods, summarizing the analysis and using these results for making judgments, decisions and predictions.

9.1 Statistical data

Collection of numerical data. It is the mathematical science that deals with the gathering, evaluation, and interpretation of numerical facts using the concept of probability of a population from examination of a random pattern.

9.1.1 Collection and tabulation of statistical data

Activity 9.2

1. Split the class into four groups. Let Group A find the last year's average result of the national exam of all subjects in the school from the school office's records. Let group B collect information about covid-19 treated in your nearest health center, hospital, or health post. Let group C measure the weight of each student in your class. Let group D collect information about the money that each student has in his or her pocket and consider its distribution based on age and sex.

   Answer the following questions using the information gathered by each group.

   a. How many students appeared for the exam?
   b. How many students scored an average mark in the regional exam?
   c. What was the score obtained by most of the students?
   d. Which groups of people are seriously infected by covid-19 diseases?
   e. What is the average weight of the class?
   f. Whose average weight is greater? Males or females?
   g. How many of the students have more than fifty Birr?

There are many definitions of the term statistics given by different scholars. However, for the purpose of this unit, we will confine ourselves to the following:

Definition 9.1

Statistics is a science of collecting, organizing, presenting, analyzing and interpreting data so that one can make a generalization.

Figure 9.1: Steps in statistics

Data collection

In Statistics, data collection is a process of gathering information from all the relevant sources to find a solution to the research problem. It helps to evaluate the outcome of the problem. The data collection methods allow a person to conclude an answer to the relevant question.

Types of data

Qualitative data and Quantitative data

Data can be classified as either qualitative or quantitative. However, statistics deals mainly with quantitative data.

Qualitative data is a means for exploring and understanding the meaning individuals or groups ascribe based on some characters whose values are not numbers, such as their color, sex, religion or the town/city they live in.

Quantitative data is a means for testing objective theories by examining the relationships among numerical variables such as scores in exams, height, weight, age or wealth. These variables can be measured, typically on instruments, so that the numbered data can be analyzed using statistical procedures.

Continuous data and discrete data

Continuous data is information that could be meaningfully divided into finer levels. It can be measured on a scale or continuum and can have almost any numeric value. For example, you can measure your height at very precise scales — meters, centimeters, millimeters and so on.

You can record continuous data at so many different measurements – width, temperature, time, and so on. This is where the key difference with discrete data lies.

Discrete data is a count that usually involves integers. For example, the number of children in a school is discrete data. You can count whole individuals. Some discrete data can be non-integer such as shoes sizes that come in halves.

There are many methods used to collect or obtain data for statistical analysis.

Here are some data collection methods or instruments:

- Primary data
- Secondary data
- Interviews
- Direct observations
- Questionnaires

Sources of collecting data

Primary Data is a data that has been generated by the researcher himself/herself, surveys, interviews, experiments, specially designed for understanding and solving the research problem at hand.

Secondary Data is data used after it is generated by large government institutions, healthcare facilities, etc. as part of organizational record keeping. The data is then extracted from more varied data files.

Interview: The main purpose of an interview as a tool of data collection is to gather data extensively and intensively. Exchanging ideas and experiences, eliciting of information pertaining to a very wide range of data in which the interviewee may wish to rehearse his past, define his present and canvass his future possibilities. Interviews can provide in-depth information pertaining to participants' experiences and viewpoints of a particular topic. Generally, interview has two types: a) In-depth and b) focus group.

Direct observations: A technique that involves systematically selecting, watching, listening, reading, touching, and recording behavior and characteristics of living things, objects, or phenomena.

Data collection surveys collect information from a targeted group of people about their opinions, behavior, or knowledge. Common types of example **surveys** are written **questionnaires**, face-to-face or telephone interviews, focus groups, and electronic (e-mail or website) surveys.

Organization of data

The process **of condensing data** and presenting it in a compact form, by putting data into a statistical table, is called **tabulation of Statistical Data**. It is a systematic presentation of numerical data in rows or/and columns according to certain characteristics. It expresses the data in a concise and attractive form which can be easily understood and used to compare numerical figures.

Presentation of data

The main purpose of data presentation is to facilitate statistical analysis. This can be done by illustrating the data using graphs and diagrams like bar graphs, histograms, piecharts, pictograms, frequency polygons, etc.

Analysis of data

In order to meet the desired purpose of investigation, data has to be analyzed. The primary objective of analyzing data is to grasp the tendency and characteristics of data from the representative values (mean, median, and mode) and from how it is distributed (dispersion), through the previous steps: organizing and presenting data.

Exercise 9.1

Classify whether the following data are quantitative or qualitative.

a. Religion
b. Amount of rain fall in a year
c. Number of academic months in a year
d. Monthly income of a person
e. Sex

Interpretation of data

Based on analyzed data, conclusions have to be drawn. This step usually involves decision making about a large collection of objects (the population) based on information gathered from a small collection of similar objects (the sample).

Some examples that show a government to take action on a certain activity.

Example 1: Information gathered about the spread of covid-19 disease in changing trends in health centers and the community.

Example 2: Statistics can be used to study the existing disease called kidney failure. It assists in determining the effectiveness of new medication and the importance of counseling.

Example 3: Demographic data about religion distribution in Ethiopia and the rate of religious growth, etc., all help policymakers in determining future needs such as providing places for mosques and churches.

Example 4: Statistical data collected on telecommunication customers provides feedback that can help to reform an appropriate service.

Example 5: Recording annual dropout rates of students in schools that help Ministry of Education of Ethiopia to reform an appropriate policy.

Population and sampling

Population in statistics refers to all over collection of individuals, objects or measurements that have a common characteristic.

Census, an enumeration of people, houses, firms, or other important items in a country or region at a particular time. The term usually refers to a population census. However, many countries take censuses of housing, manufacturing, and agriculture.

Censuses, being expensive, are taken only at infrequent intervals: every 10 years in many countries, every 5 years or at irregular intervals in other countries.

Sampling frame is the actual set of units from which a sample has been drawn: in the case of a simple random sample, all units from the sampling frame have an equal chance to be drawn and to occur in the sample.

Reaching access to the whole group (or population) is usually unmanageable, expensive and sometimes destructive. Therefore, instead of examining the whole group, a researcher examines a small part of the group, called a sample.

The following table shows scores of students in mathematics

Table 9.1: Scores of students in mathematics	
Name of student x	Score (out of 100) $S(x)$
Abdul-aziz	84
Abebe	79
Chala	81
Dendir	75
Eyasu	83
Kedija	77
Meron	80
Nardos	72
Obong	74
Yonas	82

The students are members of the population and their score, S, is the scores in mathematics.

Table 9.2: Average price of major cereal crop items in Ethiopia	
Name of food items	Price in Birr per quintal $P(x)$
Teff	4500
Barely	4800
Wheat	3800
Maize	3000
Sorghum	2500
Millets	4050
Rice	3500

The major cereal crops are members of the population and p is their price.

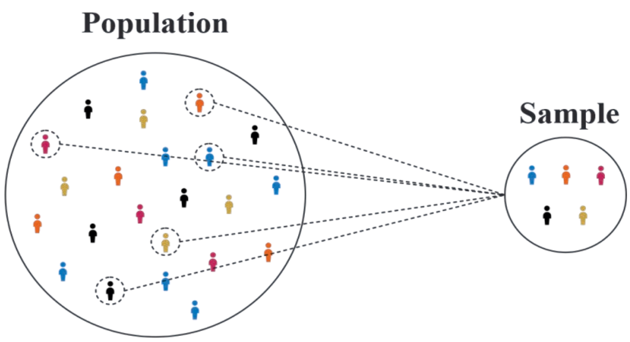


Figure 9.2: sampling

Example

What are the population and the sample in the following research situations?

a) A research randomly selected 1,000 students from different primary schools, wanting to know how much time the Ethiopian students in primary school study per day on average.

b) A factory manufactured 100, 000 machines this year. The quality control unit of the factory chooses 250 of them to evaluate the rate of defective products.

Solution:

a) The total number of primary school students is the population whereas 1,000 students among the population are the sample.

b) The population is 100,000 whereas 250 machines are the sample.

Exercise 9.2

What are the population and sample in the situation below?

a. A researcher wants to know the average income of 5,000 residents of a village. She conducted interviews to 150 people living in the village.

b. A cosmetic company implemented an advertisement that targeted all women in their 20s in Ethiopia. They randomly surveyed 1,000 women in their 20s to examine the effect of the advertisement.

9.1.2 Graphical presentations of statistical data

The first step to make a frequency distribution table is to draw two rows on a piece of paper. Then, look through your data set and list all the possible outcomes in the data in the upper rows. Use the bottom row to write for each time that particular outcome occurred in the data.

Graphical representation is another way of analyzing numerical data. A graph is a sort of chart through which statistical data are represented in the form of lines or curves drawn across the coordinated points plotted on its surface. Graphs enable us to study the cause and effect relationship between two variables. Graphs help to measure the extent of change in one variable when other variable changes by a certain amount.

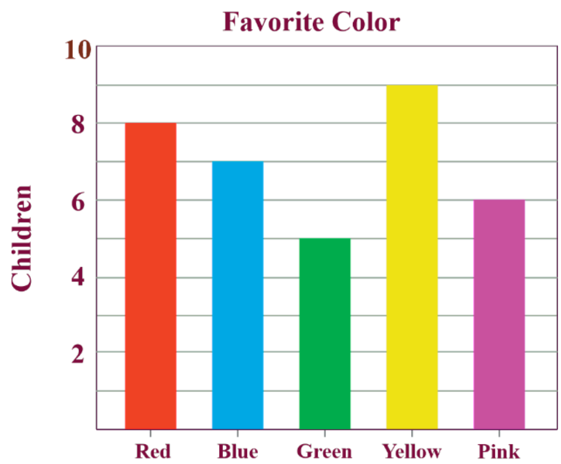


Figure 9.3: Bar chart

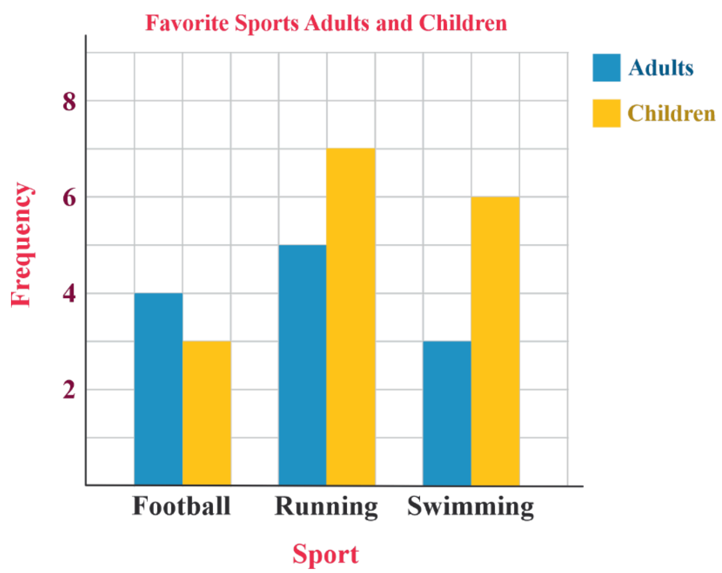


Figure 9.4: Dual Bar

Pie-charts

Data can be displayed on a pie chart-a circle divided into sectors. The size of the sector is in direct proportion to the frequency of the data. The sector size does not show the actual frequency. The actual frequency can be calculated easily from the size of the sector.

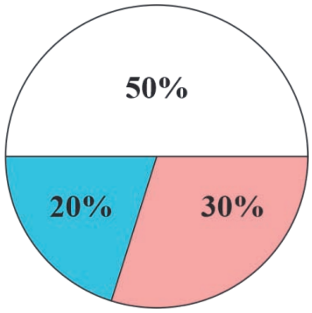


Figure 9.5: Pie-chart

Example

In a survey of 120 children, they were asked to choose their favorite color. The total 120 is represented by 100%. It follows that if 120 represent 100%, then

- **a.** 50% represents $120\left(\frac{50}{100}\right) = 60$ children whose favorite is white.
- **b.** 30% represents $120\left(\frac{30}{100}\right) = 36$ children whose favorite is purple.
- **c.** 20% represents $120\left(\frac{20}{100}\right) = 24$ children whose favorite is blue.

Methods to represent a frequency distribution:

Distributions and histograms

Data should be organized once it is collected so that it can be manageable. Data that is not organized is called raw data.

Definition 9.2

A variable V refers to a characteristic or attribute of an individual or an organization that can be measured or observed and that varies among the people or organization being studied. A variable typically will vary in two or more categories or on a continuum of scores, and it can be measured.

The drawing of tables or graphs enables us to organize raw data and help us manage the data.

Example 1

Suppose there are 12 people in a village whose weights in kilograms were measured as follows: $55, 62, 49, 67, 55, 62, 62, 49, 67, 62, 55, 49$. These data are raw. Organize the data using a table. What are the variables?

Weight in kg (V)	49	55	62	67
Number of people (f)	3	3	4	2

The above table is called the frequency distribution table. The variables are the weight and the number of people is the frequency.

Example 2

There is an age data of 20 customers who visited a shopping center. 2,4,11,12,18,22,25,28,30,30,31,34,38,39,45,48,49,50,60,64. Complete the frequency table.

Solution:

Let x be the age of the customers. Divide the data into groups of 10 years. Each interval in the frequency distribution table is called class. This type of frequency table is called grouped frequency distribution table in grade 11 Unit 7.

Class interval	frequency
$0 \leq x < 10$	2
$10 \leq x < 20$	3
$20 \leq x < 30$	3
$30 \leq x < 40$	6
$40 \leq x < 50$	3
$50 \leq x < 60$	1
$60 \leq x < 70$	2

Exercise 9.3

Complete the frequency tables with the data below.

a) 1, 1, 1, 2, 3, 3, 4, 4, 4, 4, 5, 5, 6, 6, 6

Value	frequency
1	
2	
3	
4	
5	
6	

b) 2,3,4,6,8,9,10,11,13,13,15,17

Class interval (x)	frequency
$0 \leq x < 5$	
$5 \leq x < 10$	
$10 \leq x < 15$	
$15 \leq x < 20$	

Definition 9.3

A histogram is a graphical representation of a frequency distribution in which the variable (V) is plotted on the x-axis and the frequency (f) is plotted on the y-axis.

Steps in drawing a histogram

1. Draw a horizontal line. This will be where we denote our classes.
2. Place evenly spaced marks along this line that correspond to the classes.
3. Label the marks so that the scale is clear and give a name to the x- axis.
4. Draw a vertical line just to the left of the lowest class.
5. Choose a scale for the vertical axis that will accommodate the class with the highest frequency.
6. Label the marks so that the scale is clear and give a name to the y- axis.
7. Construct bars for each class. The height of each bar should correspond to the frequency of the class at the base of the bar. We can also use relative frequencies for the heights of our bars.

Example

Plot the following mathematics scores (s) by a histogram

Class interval	frequency
$25 \le s < 30$	2
$30 \le s < 35$	2
$35 \le s < 40$	5
$40 \le s < 45$	10
$45 \le s < 50$	6
$50 \le s < 55$	2
$55 \le s < 60$	3

Solution:

In this graph, we shall take class intervals on the x- axis and frequencies in the y-axis. Before plotting the graph, we have to convert the class into their exact limits. Histogram plotted from the data

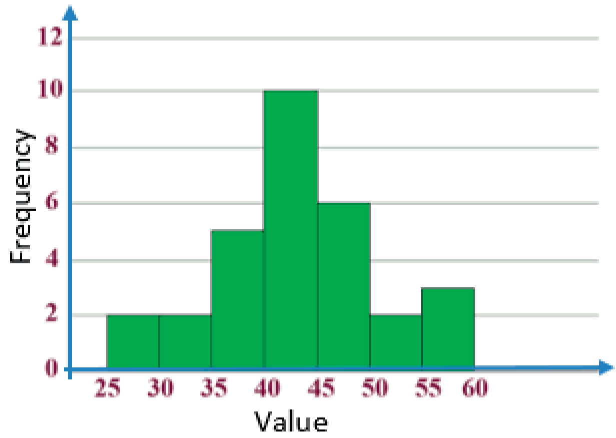


Figure 9.6: Histogram

Activity 9.3

A group of students collected the data on the height in cm of one type of plant. Here is the data of 30 pieces of the plant.

61, 63, 64, 66, 68, 69, 71, 71.5, 72, 72.5, 73, 73.5, 74, 74.5, 76, 76.2, 76.5, 77, 77.5,

Construct a frequency distribution table and a histogram for these data.

Exercise 9.4

1. Construct a histogram for the following frequency distribution table that describes the frequencies of weights of 25 students in a class.

Weight(w) in kgs	Frequency (number of students)
$45 \leq w < 50$	4
$50 \leq w < 55$	10
$55 \leq w < 60$	8
$60 \leq w < 65$	3

2. Complete the frequency distribution table and draw the corresponding histogram for the class interval by 5cm from 145cm. The row data is given as follows: 145, 148, 150, 157,160, 164, 166, 169, 171

Height(x)	Frequency (number of students)
$145 \leq x < 150$	
$150 \leq x < 155$	
$155 \leq x < 160$	
$160 \leq x < 165$	
$165 \leq x < 170$	
$170 \leq x < 175$	

9.1.3 Measures of central tendency

A measure of central tendency is a single value that attempts to describe a set of data by identifying the central position within that set of data. As such, measures of central tendency are sometimes called measures of central location. They are also classed as summary statistics. The mean (often called the average) is most likely the measure of central tendency that you are most familiar with, but there are others, such as the median and the mode.

The mean, median and mode are all valid measures of central tendency, but under different conditions, some measures of central tendency become more appropriate to use than others. In the following sub-units, we will look at the mean, mode and median, and learn how to calculate them and under what conditions they are most appropriate to be used.

Mean

Activity 9.4

1. A teacher gave a mid-term exam to 20 students. The full score of this exam is 25. The students scored the following:

   9,9,9,10,11,11,12,12,12,12,13,14,14,14,14,14,18,18,18,21

   a. Calculate the mean score of the students.

   b. How many students score below the mean, exactly the mean and above the mean?

   c. How well the students did the exam?

Definition 9.4

The Arithmetic mean (or average) is the most popular and well-known measure of central tendency. It can be used with both discrete and continuous data, although its use is most often with continuous data (see our Types of Variable guide for data types). The mean is equal to the sum of all the values in the data set divided by the number of values in the data set. So, if we have n values in a data set and they have values $x_1, x_2, \ldots \ldots . x_n$ the mean, usually denoted by $\bar{x}$ (pronounced " x bar"), is:

$$\bar{x} = \frac{x_1 + x_2 + \cdots + x_n}{n} .$$

The mean is essentially a model of your data set. It is the value that is most common. You will notice, however, that the mean is not often one of the actual values that you have observed in your data set. However, one of its important properties is that it minimizes error in the prediction of any one value in your data set. That is, it is the value that produces the lowest amount of error from all other values in the data set.

The mean can also be calculated from its frequency distribution. So, if the values $x_1, x_2, \ldots \ldots . x_n$ occur $f_1, f_2, f_3, \ldots \ldots f_n$ times, respectively, then the mean ($\bar{x}$) is given by

$$\bar{x} = \frac{f_1 x_1 + f_2 x_2 + f_3 x_3 + \cdots \ldots x_n f_n}{f_1 + f_2 + f_3 + \cdots \ldots + f_n}$$

Example 1

Seven children have the following number of textbooks: 5, 4, 6, 4, 3, 6 and 7.

Calculate the mean of the textbooks of the children.

Solution:

$$\bar{x} = \frac{5+4+6+4+3+6+7}{7} = \frac{35}{7} = 5$$

Example 2

Calculate the mean test score from the frequency distribution table below that shows the test scores of 10 students.

Test scores	Frequency
10	0
20	4
30	1
40	3
50	2

Solution:

$$\bar{x} = \frac{10(0)+20(4)+30(1)+40(3)+50(2)}{10} = \frac{330}{10} = 33$$

Example 3

Ten people have the following number of pencils: 6, 5, 6, 5, 3, 6, 6, 10, 10 and 3.

Calculate the mean of the pencils of the people.

Solution:

No. of pencils (x)	Frequency (f)
3	2
5	2
6	4
10	2

$$\bar{x} = \frac{3(2)+5(2)+6(4)+10(2)}{10} = \frac{60}{10} = 6$$

Exercise 9.5

a) Calculate the mean of the weights in kg of 6 people below

45, 50, 55, 60, 70, 80

b) Calculate the mean height of the people shown in the frequency distribution below.

Height (cm)	Frequency
160	3
170	2
175	1

Properties of mean

Activity 9.5

Six students have the following amount of money: Birr 55, 45, 60, 40, 30 and 70.

a. Calculate the mean of money in the group.
b. Find the sum of the differences of each value from the mean.
c. If one gives Birr 5 to each student, calculate the new mean.
d. If each student gives Birr 5 to somebody from what they have, calculate the new mean.
e. If each of their money is multiplied by 5, what is the new mean?
f. Discuss the answers you got in a, b, c, d and e

Property 1: The sum of the deviations from the mean is zero.

If $\bar{x}$ is the arithmetic mean of n observations $x_1, x_2, \ldots \ldots . x_n$, then

$$(x_1 - \bar{x}) + (x_2 - \bar{x}) + (x_3 - \bar{x}) \ldots \ldots . . +(x_n - \bar{x}) = 0$$

Proof: $\bar{x} = \frac{x_1 + x_2, \dots\dots + x_n}{n}$

$$x_1 + x_2 + \cdots \dots . + x_n = n\bar{x} \dots\dots\dots\dots\dots\dots\dots *$$

$(x_1 - \bar{x}) + (x_2 - \bar{x}) + (x_3 - \bar{x}) \dots\dots . . + (x_n - \bar{x}) = x_1 + x_2, \dots\dots . + x_n - n\bar{x} = n\bar{x} - n\bar{x} = 0, \dots\dots\dots\dots . . Using *$

Hence, $(x_1 - \bar{x}) + (x_2 - \bar{x}) + (x_3 - \bar{x}) \dots\dots . . + (x_n - \bar{x}) = 0$

Property 2:

The mean of n observations $x_1, x_2, \dots\dots . x_n$ is $\bar{x}$. If each observation is increased by p, the mean of the new observations is $(\bar{x} + p)$.

Proof: $\bar{x} = \frac{x_1 + x_2 +, \dots\dots + x_n}{n}$

$$x_1 + x_2 + \cdots \dots . + x_n = n\bar{x}$$

The mean of $(x_1 + p), (x_2 + p), (x_3 + p) \dots\dots , (x_n + p)$

$$= \frac{(x_1+p) + (x_2+p) + (x_3+p) + \cdots \dots +, (x_n+p)}{n}$$

$$= \frac{(x_1 + x_2 + \cdots \dots + x_n) + np}{n} = \frac{x_1 + x_2 + \cdots \dots + x_n}{n} + \frac{np}{n} = \bar{x} + p$$

Hence, the mean of the new observations is $\bar{x} + p$.

Property 3:

The mean of n observations $x_1, x_2, \dots\dots . x_n$ is $\bar{x}$. If each observation is decreased by p, the mean of the new observations is $\bar{x} - p$.

Property 4:

The mean of n observations $x_1, x_2, \dots\dots . x_n$ is $\bar{x}$. If each observation is multiplied by a nonzero number p, then the mean of the new observations is $p\bar{x}$.

$$\bar{x} = \frac{x_1 + x_2, \dots\dots + x_n}{n}$$

$$x_1 + x_2 + \cdots \dots . + x_n = n\bar{x}$$

The mean of $x_1p,\ x_2p, \dots\dots x_np$ $= \frac{(x_1p)+(x_2p)+,\dots\dots\dots+(x_np)}{n}$

$$= \frac{(x_1 + x_2 + \cdots\dots + x_n)p}{n} = p\bar{x}.$$

Property 5:

The mean of n observations $x_1,\ x_2, \dots\dots x_n$ is $\bar{x}$. If each observation is divided by a nonzero number p, then the mean of the new observations is $\frac{\bar{x}}{p}$.

Exercise 9.6

1. If $\bar{x} = 4$, find the new mean ín case
   a) each value increases by 1.
   b) each value decreases by 2.
   c) each value is multiplied by 3.
   d) each value is divided by 2
2. The weekly mean number of customers who use a bus stop is 100 per day. If 10 more customers use it every day, what is the weekly mean number of the customers?
3. A car factory manufactures 1,000 cars every day on average. One day a new machine is installed, and they can now produce twice as many cars every day. What is the mean number of cars that they produce every day?

Median

Definition 9.5

The median is the middle score for a set of data that has been arranged in order of magnitude. The median is less affected by extreme values and skewed data.

Example 1

The following are the scores of 11 people. Find the median.

65, 55, 89, 56, 35, 14, 56, 55, 87, 45, 92

Solution:

We first need to rearrange that data into order of magnitude (smallest first):

14, 35, 45, 55, 55, **56**, 56, 65, 87, 89, 92

Our median mark is the middle mark - in this case, **56** (highlighted in bold). It is the middle mark because there are 5 scores before it and 5 scores after it. This works fine when you have an odd number of scores, but what happens when you have an even number of scores? What if you had only 10 scores? Well, you simply have to take the middle two scores and average the result. So, if we look at the example below:

65, 55, 89, 56, 35, 14, 56, 55, 87, 45

We again rearrange that data into order of magnitude (smallest first):

14, 35, 45, 55, **55**, **56**, 56, 65, 87, 89

Now, we have to take the 5^{th} and 6^{th} scores in our data set and average them to get a median of 55.5.

$$\text{Median} = \frac{55+56}{2} = 55.5$$

Properties of the median

1. It is not affected by extreme values.
2. It is unique for a given data set.

Example 2

Find the median of the numbers $5, 3, 99, x, 4$ where x is greater than 10 less than 99.

Solution:

Arranging in increasing order as $3, 4, 5, x, 99$, hence, 5 is the median and is not affected by extreme value, 99 and is unique.

Exercise 9.7

a) Find the median of the data below.

5, 7, 3, 10, 1, 5, 9

b) Calculate the mean and median and compare the values of the two for the data below.

15, 10, 2, 6, 7, 20, 3, 18, 100

Mode

Definition 9.6

The **mode** is the most frequent value in a set of data.

Example 1

Find the mode of the following data.

5, 10, 20, 20, 30, 35, 35, 35, 40, 50, 70, 70

Solution:

The most frequent value is 35. Hence, it is the mode.

Example 2

A teacher gave a mid-term exam to 20 students. The full score of this exam is 25%. The students scored the following:

18, 9, 11, **14**, 11, **12**, 13, 13, **12**, 9, 10,
14, **12**, **14**, **12**, **14**, 18, 18 , 21, 9

Solution:

The most frequent value in the data are **14** and **12.** Therefore the mode are 14 and 12. On a histogram, mode represents the highest bar in a bar chart or histogram. You can, therefore, sometimes consider the mode as being the most popular option. An example of a mode is presented below:

Example 3

Find the mode of the following data. 1, 3, 6, 7, 9, 8, 10, 17, 18

Solution:

Here, there is no most frequent value. No value repeats itself. Hence, there is no mode.

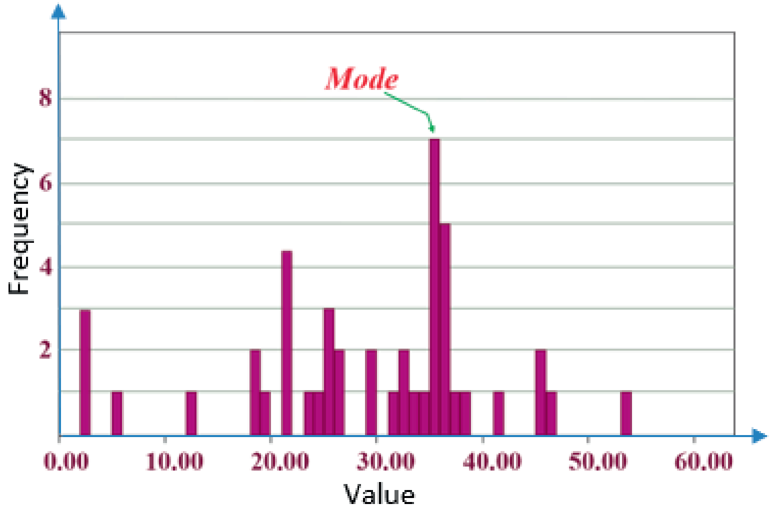


Figure 9.7: Mid exam scores of grade 9 students

Normally, the mode is used for categorical data where we wish to know which is the most common category, as illustrated below:

However, one of the problems with the mode is that it is not unique, so it leaves us with problems when we have two or more values that share the highest frequency, such as below.

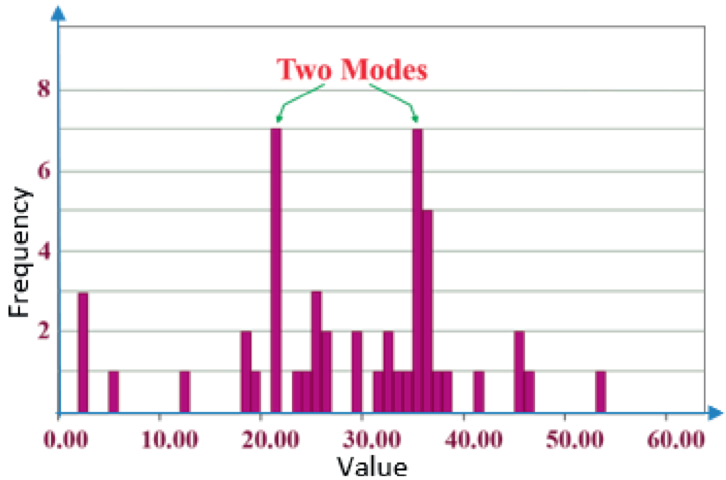


Figure 9.8: The final –exam score of grade 9 students

Properties of the mode

1. The mode is not always unique
2. The mode can also be used for qualitative data

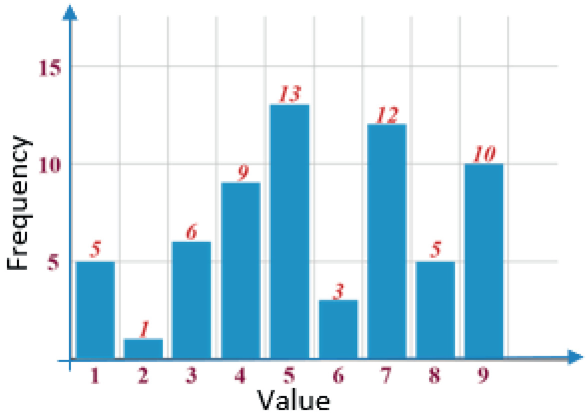
Exercise 9.8

Find the mode of the following data.

a) 2, 3, 3, 4, 5, 5, 5

b) 13, 10, 16, 15, 12, 14, 13, 16

c) Find the mode from the bar chart on the right



Exercise 9.9

1. A boy recorded the number of book pages that he read over the last 5 days. 24, 10, 16 ,7, 33
   a. Find the mean number of pages read
   b. Find the median
   c. Find the mode
2. A woman was recording her weight in kg in the last six months as follows: 88, 86, 89, 91, 85, 90
   a. What is the mean weight of the woman in the last six months?
   b. Find the median
3. Is it possible to find the arithmetic mean of qualitative data?
4. Record car accidents in your neighboring city/town to your school together with their teacher.
   a. Find the mean car accident in the city/town

b. Find the median

c. Find the most frequent (highest frequency) in the accident

5. Find the mean, median and mode of the following set of numbers:

$82, 23, 59, 94, 70, 26, 32, 83, 87, 94$, and 32

6. Answer the following by referring to the frequency distribution given.

a. Find the mean, median and mode

b. How many of the values are greater than or equal to 2?

V	-4	-1	0	1	2	3
f	2	3	4	5	3	3

7. If the mean is 5 for the data $2, 3, x, 5, 6, 12$, find the value x.

8. If the mean of a, b, c, d is k, then what is the mean of $a + b, 2b, c + b, b + d$?

9. If the mean of $y + 2, y + 4, y + 6$ and $y + 10$ is 13, find the value of y.

10. In an examination, the mean of marks scored by a class of 40 students was calculated as 72.5. Later on, it was detected that the marks of one student were wrongly copied as 48 instead of 84. Find the correct mean.

11. The mean monthly salary of 12 employees of a firm is Birr 1450. If one more person joins the firm who gets Birr 1645 per month, what will be the mean monthly salary of 13 employees?

9.1.4 Measures of dispersion

When comparing sets of data, it is useful to have a way of measuring the scatter or spread of the data.

Activity 9.6

Consider the following four sets of numbers representing the ages of children

Group	Values										Mean
I	5	7	6	7	4	5	4	6	8	5	
II	2	1	4	3	9	6	3	2	4	3	
III	6	5	7	5	6	7	5	6	5	7	
IV	5	5	5	5	5	5	5	5	5	5	

1. Complete the table by finding the sum of each group and by calculating the mean.
2. Are the means equal?
3. Compare the difference between the mean and each observed value in Group I, II, III and IV.
   i. Which group's mean is closest to each value?
   ii. Which group has the greatest difference between the mean and each data value?
4. Compare the variation of each group?
   i. Which group shows most variation?
   ii. Which group shows no variation?
   iii. Which group shows slight variation?
5. Find the range for each group.

Dispersion or Variation is the scatter (or spread) of data values from a measure of central tendency. There are several measures of dispersion that can be calculated for a set of data. In this sub-unit, we will consider only three of them, namely, the range, the variance and the standard deviation.

1. Range

Activity 9.7

Given the raw data as follows: 6, 8, 2, 10, 5, 8, 11, 7. Find the difference between the largest and the smallest.

The simplest and the crudest measure of the dispersion of quantitative data is the range

Definition 9.7

The difference between the largest and smallest values of a set of numerical data is called the **range R** .i.e. Range = Largest value − Smallest value.

Example 1

The scores of 10 students in mathematics test is as follows: 6, 7, 5, 8, 3, 8, 9, 5, 4, 5.5. Find the range of the scores of the students.

Solution:

Range = Largest value – Smallest value = 9 − 3 = 6

Example 2

Find the range of the frequency distribution from the table below:

V	1	2	4	5	6
f	3	6	8	2	1

Solution:

Range = Largest value – Smallest value = 6 − 1 = 5

Exercise 9.10

a) The weights of a family member composed of 5 persons are as follows: Find the range. 35,40,85,65,70

b) The table below shows the number of customers who came to a shop in the week. Find the range.

Day	Mon	Tue	Wed	Thu	Fri	Sat	Sun
Number of customer	30	23	45	17	5	41	54

2. Variance and standard deviation

The activities below assist you to learn how to find variance and standard deviation.

Activity 9.8

Consider the following numerical data: $4, 5, 7, 8, 7, 5$

a. Find the mean

b. Find the difference of each value from the mean (deviation)

c. Square each of the deviations

d. Calculate the mean of these squared deviations and its principal square root

Definition 9.8

Variance is the mean of the squared deviations of each value from the arithmetic mean and is denoted by σ^2

Definition 9.1

Standard deviation is the square root of variance.

How to calculate standard deviation can be summarized as the following steps:

1. Calculate the arithmetic mean $\bar{x}$ of the data
2. Find the deviation of each data from the mean $(x - \bar{x})$
3. Calculate the square of each of these deviations, $(x - \bar{x})^2$
4. Find the mean of these squared deviations. This value is the variance (σ^2). Let n be the number of observations or cases
5. $\sigma^2 = \frac{(x_1-\bar{x})^2+(x_2-\bar{x})^2+(x_3-\bar{x})^2+\cdots+(x_n-\bar{x})^2}{n}$
6. Take the principal square root of the variance, (σ^2), i.e.

   Standard deviation $(\sigma) = \sqrt{\text{variance}}$,

   $\sigma = \sqrt{\frac{(x_1-\bar{x})^2+(x_2-\bar{x})^2+(x_3-\bar{x})^2+\cdots+(x_n-\bar{x})^2}{n}}$.

Example 1

Calculate the variance, σ^2 and standard deviation, σ if the number of televisions sold in each day of a week is 13, 8, 4, 9, 7, 12, 10.

Solution:

$\bar{x} = \frac{13+8+4+9+7+12+10}{7} = \frac{63}{7} = 9$

x_i	$x_i - \bar{x}$	$(x_i - \bar{x})^2$
13	**4**	**16**
8	**-1**	**1**
4	**-5**	**25**
9	**0**	**0**
7	**-2**	**4**
12	**3**	**9**
10	**1**	**1**

Variance, $\sigma^2 = \frac{16+1+25+0+4+9+1}{7} = \frac{56}{7} = 8$

Standard deviation, $\sigma = \sqrt{\sigma^2} = \sqrt{8} \cong 2.83$

Exercise 9.11

1. With the data 1, 3, 5, 5, 7, 9
   a. Calculate the mean
   b. Calculate the variance
   c. Calculate the standard deviation
2. With the data 5, 2, 3, 3, 7 and their arithmetic mean, $\bar{x} = 4$
   a. Find the variance
   b. Find the standard deviation

Example 2

48 students were asked to write the total number of hours per week they spent watching television. With this information, find the standard deviation of hours spent watching television

Solution:

Hours (h) per week	6	7	8	9	10	11	12
f	3	6	9	13	8	5	4

$$\bar{x} = \bar{x} = \frac{6\cdot3+7\cdot6+8\cdot9+9\cdot13+10\cdot8+11\cdot5+12\cdot4}{48}$$

$$= \frac{432}{48} = 9$$

x_i	f_i	$x_i f_i$	$x_i - \bar{x}$	$(x_i - \bar{x})^2$	$f_i(x_i - \bar{x})^2$
6	3	18	-3	9	27
7	6	42	-2	4	24
8	9	72	-1	1	9
9	13	117	0	0	0
10	8	80	1	1	8
11	5	55	2	4	20
12	4	48	3	9	36
	$n = 48$				

$$\sigma^2 = \frac{27 + 24 + 9 + 0 + 8 + 20 + 36}{48} = \frac{124}{48} = \frac{31}{12}$$

Hence, $\sigma = \sqrt{\sigma^2} = \sqrt{\frac{31}{12}} \cong \sqrt{2.58} \cong 1.6$

Exercise 9.12

A student asked his 20 classmates how many books they read every month.

Number of books	1	2	3	4
Frequency(f)	6	9	4	1

Complete the table and find the variance and the standard deviation.

x_i	f_i	$x_i f_i$	$x_i - \bar{x}$	$(x_i - \bar{x})^2$	$f_i(x_i - \bar{x})^2$
1	6	6			
2	9	18			
3	4	12			
4	1	4			
	$n = 20$				

Properties of standard deviation

Activity 9.9

Consider the following data which shows the amount of rain fall in mm in the last five days. $20, 30, 50, 70, 80$

1. Find the mean.
2. Calculate the variance and standard deviation.
3. In the next five days, if the rainfall increases by 5mm each, i.e. $25, 35, 55, 75, 85$.
   a. Find the mean of rainfall in those five days.
   b. Find the variance and standard deviation of rainfall in those five days.
   c. Compare the answers with those found in 1 and 2 above.
   d. Discuss the comparison you found in c.

The above group work assists you in noticing the following properties:

1. Property 1.

If a constant k is added to each value of a data, then the new variance is the same as the old variance. The new standard deviation is also the same as the old.

Proof:

Let $x_1, x_2, \dots\dots x_n$ be n observations with mean $\bar{x}$ and variance σ^2. Adding to each observation, we get $x_1 + k,\ x_2 + k, \ldots \ldots x_n + k$, then the new mean is $\bar{x} + k$.

New variance $= \frac{[(x_1+k)-(\bar{x}+k]^2+[(x_2+k)-(\bar{x}+k]^2+\cdots+[(x_n+k)-(\bar{x}+k]^2}{n}$

$= \frac{[x_1-\bar{x}]^2+[x_2-\bar{x}]^2+\cdots+[x_n-\bar{x}]^2}{n} = \sigma^2$ and the new standard deviation is σ

Example

Consider a set of data 2, 3, 4, 3

a. Calculate the variance and standard deviation.

b. Add 3 to each value, and then find the new variance and the new standard deviation.

Solution:

$$\bar{x} = \frac{2+3+4+3}{4} = 3$$

$$x - \bar{x}: -1, 0, 1, 0 \text{ and } (x - \bar{x})^2: 1, 0, 1, 0$$

a. $\sigma^2 = \frac{1+0+1+0}{4} = \frac{2}{4} = 0.5$ & $\sigma = \sqrt{0.5} \cong 0.7071$

b. Adding 3: we have the data $5, 6, 7, 6$

Let $\bar{x}_1$ be the original mean and $\bar{x}_2$ be the new mean.

$\bar{x}_2 = \frac{5+6+7+6}{4} = \frac{24}{4} = 6$

Or using Property 2 of mean,

$\bar{x}_2 = \bar{x}_1 + \text{p} = 3 + 3 = 6$ $\quad x - \bar{x}: -1, 0, 1, 0$ and $(x - \bar{x})^2: 1, 0, 1, 0$

The new variance is $\sigma^2 = \frac{1+0+1+0}{4} = \frac{2}{4} = 0.5$ & $\sigma = \sqrt{0.5} \cong 0.7071$

The new variance = the old variance and the new standard deviation = the old standard deviation.

Exercise 9.13

a) Find the variance and standard deviation of the following data. $3, 9, 2, 7, 4$

b) If you add 2 to each value of a), how will the values of variance and standard deviation change?

c) If you subtract 2 from each value of a), how will the values of variance and standard deviation change?

Property 2.

If each value of data is multiplied by a constant c, then

a. The new variance is c^2 times the old variance

b. The new standard deviation is $|c|$ times the old standard deviation.

Proof:

Consider n observations$x_1, x_2, \ldots \ldots . x_n$, the variance $\sigma^2$and the mean $\bar{x}$.

Multiplying each data value by c, we have $c\bar{x}$. The new variance

$$= \frac{(cx_1 - c\,\bar{x})^2 + (cx_2 - c\,\bar{x})^2 + \cdots \ldots\ldots\ldots + (cx_n - c\,\bar{x})^2}{n}$$

$$= \frac{c^2(x_1 - \bar{x})^2 + c^2(x_2 - \bar{x})^2 + \cdots \ldots \ldots \ldots . + c^2(x_n - \bar{x})^2}{n}$$

$$= c^2 \text{ times the old variance} = c^2\sigma^2$$

Therefore, the new standard deviation $= |c|\sigma$

Exercise 9.14

a. Find the variance and standard deviation of the following data: $3, 16, 1, 12$

b. Find the new variance and standard deviation after multiplying each value by 2

c. Find the new variance and standard deviation after dividing each value by 7.

Exercise 9.15

1. Find the mean, median, and mode(s) of the following data $3, 4, 5, 5, 7, 8, 9, 9, 10$
2. Find the range, variance and standard deviation of the following data.

   5, 5,2,5,1, 4,4,3,6,5,3,5.
3. Find the range, variance and standard deviation of the distribution in the table below:

V	-3	-2	-1	0	1	2	3
f	3	2	1	8	1	2	3

4. If the standard deviation of the data $5, y,\ 5, 5, 5$ *is* 5, what is the value of y?
5. If the variance of a, b, c, d, e *is* m , then calculate
   a. The variance of $2a, b + a, c + a, d + a$ *and* $e + a$.
   b. The standard deviation of $2a, b + a, c + a, d + a$ *and* $e + a$.
   c. The standard deviation of $a^2, ab,\ ac,\ ad$ *and* ae.
6. The daily rainfall for the last five days was $60, 90, 150, 210, 240$. For the next five days, the amount of rainfall was doubled each, i.e., $120, 180, 300, 420,$ *and* 480.
   a. Calculate the mean, variance and standard deviation of the doubled rain fall.
   b. Compare the above result $(6a)$ with questions number 1 and 2 of activity 9.9.
7. Calculate the variance, σ^2 and standard deviation, σ if the number of exercise books sold in each day of a week is 9, 8, 4, 12, 5, 14, 11.
8. Twenty students were asked to write the total number of hours per week they spent watching Ethiopian television. With this information find the standard deviation of hours spent watching Ethiopian television.

Hours (h) per week	4	3	2	1
f	2	6	5	7

9.2 Probability

Introduction

In your grade 8 lessons, you came across with the word probability as you often apply it," The probability of passing an exam'', or "there is a high probability that they can win the game", and so on. Here, the concept probability describes guesses of possibilities. Recall what you have discussed in grade 8 on this lesson based on the group work below:

Activity 9.10

1. Which of the following is different from the others?

   a. Chance b. Interpretation c. Possibilities d. Uncertainty.

2. All the possible outcomes that can occur when a coin is tossed twice are listed on a paper: HH, HT, TH, TT

   What is the probability of having at least a head?

Probability is a numerical value that describes the likelihood of the occurrence of an event in an experiment.

Definition 9.10

An experiment is a trial by which an observation is obtained but whose outcome canot be predicted in advance.

Experimental probability is a probability determined by using data collected from a repeated data.

Definition 9.11

If an experiment has n equally likely outcomes and if m is a particular outcome, then the probability of this outcome occurring is $\frac{m}{n}$

Example

All the possible outcomes that can occur when a coin is tossed twice are HH, HT, TH, TT. What is the probability of having

a) Exactly one tail?

b) At least one head?

Solution:

a) Two (HT & TH) out of the four possibilities. Hence, probability $= \frac{2}{4} = \frac{1}{2}$

b) Three (HH, HT & TH) out of the four possibilities. Hence, probability $= \frac{3}{4}$

Exercise 9.16

1. All the possible outcomes that can occur when a coin is tossed twice are HH, HT, TH, TT. What is the probability of having
   a) Exactly one head?
   b) No tail?
2. List all the possible outcomes.
   a) A fair coin is tossed once
   b) A fair coin is tossed three times

Sample space and event

Definition 9.13

Sample space for an experiment is the set of all possible outcomes of an experiment.

Example 1

a. List the sample space in tossing one coin.

b. Give the sample space in throwing one die.

c. What is the sample space in throwing two coins?

Solution:

a. In tossing one coin, there are only two possibilities: Heads (H) or Tails (T) . Therefore, $S = \{H, T\}$.

b. In throwing one die, any one of the numbers $1, 2, 3,4,5,6$ will appear on the upper face of the die. Hence, $S = \{1,2,3,4,5,6\}$.

c. The set of possible outcomes is $S = \{HH, HT, TH, TT\}$.

Definition 9.14

An **event** is a subset of a sample space set or a sample space.

Activity 9.11

Duguma throws a fair die once. According to this experiment, discuss the following

Figure 9.9: a die

1. Is it possible to guess the number that shows on the upper face of the die? Why?
2. Write the sample space.
3. From the experiment, give an example of an event.
4. Which event (a) or (b) is certain and which is impossible?
   a) The number on the upper face of the die is 8.
   b) The number on the upper face of the die is a natural number.
5. Determine the possibilities of the following events.
   i. The number on the upper face of the die is 6
   ii. The number on the upper face of the die is 0
   iii. The number on the upper face of the die is less than or equal to 4
6. Discuss the following words:
   a. Experiment
   b. Sample space
   c. Event
   d. Impossile event
   e. Certain event

Example 2

The numbers 4 to 30 are each written on one of 27 identical cards. One card is chosen at random.

i. List the elements of all the possible outcomes

ii. Write all the possible elements of the following events

- **a.** the number is greater than or equal to 4
- **b.** the number is less than or equal to 30
- **c.** the number is greater than 15
- **d.** the number is divisible by 3
- **e.** the number is even

Solution:

i. $S = \{4, 5, 6, \ldots \ldots .. 30\}$

ii. Write all possible elements of the following events:

- **a.** $S = \{4, 5, 6, \ldots \ldots .. 30\}$
- **b.** $S = \{4, 5, 6, \ldots \ldots .. 30\}$
- **c.** $S = \{16, 17, 18 \ \ldots \ldots .. 30\}$
- **d.** $S = \{6, \ 9, 12, \ldots \ldots .. 30\}$
- **e.** $S = \{4, 6, \ 8 \ldots \ldots .. 30\}$

Exercise 9.17

The numbers 3 to 30 are each written on one of 28 identical cards. One card is chosen at random.

i. List the set of all possible outcomes

ii. Write all the possible elements of the following events

- a) the number is greater than or equal to 5
- b) the number is less than or equal to 15

Probability of an event

Activity 9.12

A coin is tossed three times, six times, nine times and twelve times and the observation is recorded as follows:

	Number of tosses				Total
The number of times the coin is tossed	3	6	9	12	30
The number of times the coin shows up heads	1	4	4	6	15
The number of times the coin shows up tails	2	2	5	6	15

What proportion of the number of tosses shows Heads? Tails? What is the probability that the outcome is Head? Tail?

If you toss a coin 10 times and get a head 6 times and a tail 4 times, then you would say that in a single toss of a coin, the probability of getting a head is $\frac{6}{10} = 0.6$. Once again, if you toss a coin 1,000 times and get a head 600 times and a tail 400 times, then you would say that in a single toss of a coin, the probability of getting a head is $\frac{600}{1{,}000} = 0.6$. We may obtain different probabilities for the same event from various experiments. Repeating the experiment sufficientely large number of times, however, the relative frequency of an outcome will tend to be close to the theoretical probability of that outcome.

Example 1

Toss a coin 10,000 times and you obtain 5010 heads,

i. If the event was tails, how many times did this event occur?

ii. According to the experiment, what was the probability of tails?

Solution:

i. The times you get a tail is:

10000 – 5010 = 4990

4990 times

ii. 4990/10000 = 0.499

Probability of tails is 0.499

Exercise 9.18

a. Toss a coin 10,000 times and you obtain 5610 heads.

i. If the event was tails, how many times did this event occur?

ii. According to the experiment, what was the probability of tails?

b. You toss a die 100 times, and each number appeared as follows:

1. 20 times
2. 16 times
3. 15 times
4. 14 times
5. 18 times
6. 17 times

i) If the event is 1, how many times did the event occur?

ii) According to the experiment, what is the probability of getting 2?

iii) What is the probability of getting 5 or 6?

iv) What is the probability of getting an odd number?

Example 2

In an experiment of choosing workers at random, a supervisor found the following result after 100 trials.

Gender of worker	Men	Women	Total
Number of workers	45	55	100

What is the probability that a randomly selected worker is a man?

Solution:

The probability that a randomly selected worker is a man will be the ratio of the number of men to the total number of trials.

The probability of a man selected $= \frac{45}{100}$

Exercise 9.19

1. In a class, there are 20 boys out of 40 classmates in total. In an experiment of choosing a classmate at random, what is the probability of choosing a boy?
2. There are 12 males and 8 females in a room. If one person is randomly selected, what is the probability that the selected one is a female?
3. A bag contains 5 red balls, 2 blue balls, and 3 black balls. Find the probability of getting a black ball if you pick a ball from the bag at random.
4. Suppose you are planning a research for public and private schools and wants to sample 1 school at random. If the ratio of the number of public schools to that of private schools is 9:1, what is the probability of sampling a private school?

Tree diagram

When more than two combined events are being considered, two-way tables cannot be used and therefore another method of representing information diagrammatically is needed. Tree diagrams are a good way of doing this. A tree diagram is one way of showing the possible outcomes of a repeated experiment.

Example 1

In an experiment of tossing three coins,

a. What are the possible outcomes?

b. How many different possible outcomes are there?

c. What is the probability of the coin facing up with

  i. Exactly two heads?

ii. Exactly one tail?

iii. Three tails

iv. At least two heads?

Solution:

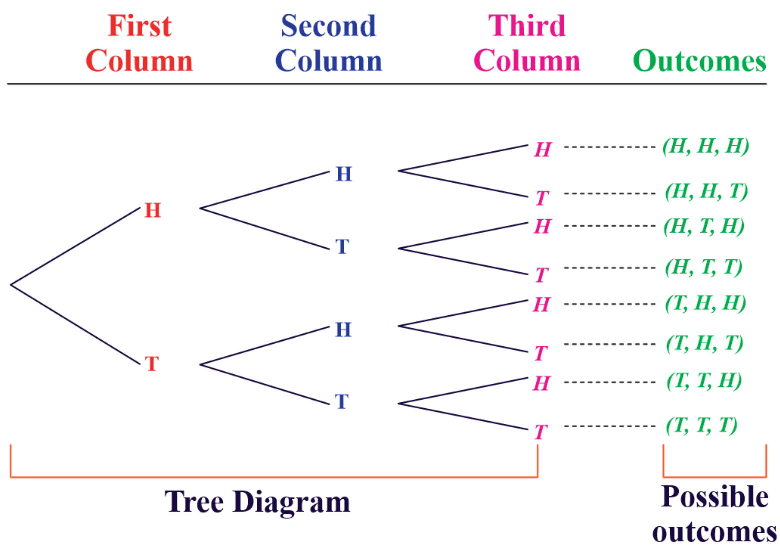


Figure 9.10: A tree diagram in tossing three coins

a. S = {HHH, HHT, HTH, HTT, THH, THT, TTH, TTT}.

b. As can be seen from a) there are 8 possible outcomes.

c. The probability of the coin facing up with is:

i. Exactly two heads $= \frac{3}{8}$.

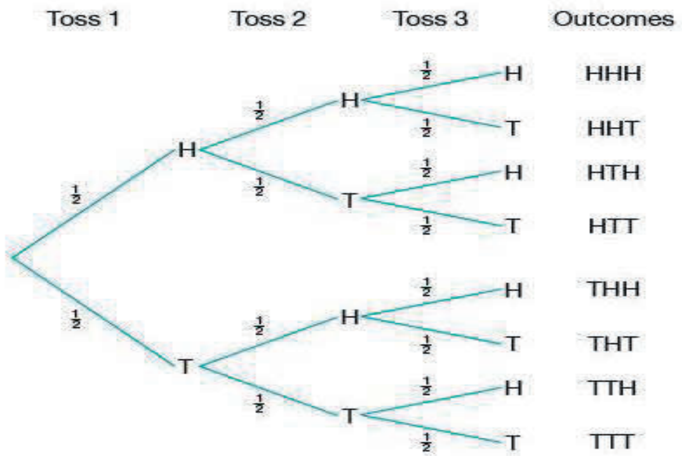
ii. Exactly one tail $= \frac{3}{8}$.

iii. Three tails $= \frac{1}{8}$.

iv. At least two heads$= \frac{4}{8} = \frac{1}{2}$.

Example 2

If a coin is tossed three times, the tree diagram is as follows. Each of the probabilities is indicated at the side of the branches.



Exercise 9.20

Calculate the probability of getting the following events when three fair coins are tossed by drawing a tree diagram.

a. Exactly one head
b. Three heads
c. At least two tails

Theoretical probability

Note that it is not always possible to perform an experiment and calculate probability. In such cases, we should have another method to calculate the probability of an event. We will see a theoretical approach to find such probabilities.

Definition 9.15

The theoretical probability of an event E, written as $P(E)$ is defined as follows:

$$P(E) = \frac{\text{Number of outcomes favourable to the event E}}{\text{Total number of possible outcomes(S)}} = \frac{n(E)}{n(S)}$$

Example 1

If you throw a die once, what is the probability that an odd number will show on the upper face of a die?

Solution:

Let the event of odd number be E

$S = \{1,2,3,4,5,6\}$, $E = \{1,3,5\}$

$$P(E) = \frac{\text{Number of outcomes favourable to the event E}}{\text{Total number of possible outcomes(S)}} = \frac{3}{6} = \frac{1}{2}$$

Example 2

A coin and a die are thrown together.

a. Sketch a tree diagram showing the outcomes of this experiment.

b. What is the probability of getting a tail and an odd number?

c. What is the probability of getting a head and a number less than or equal to 5?

Solution:

a.

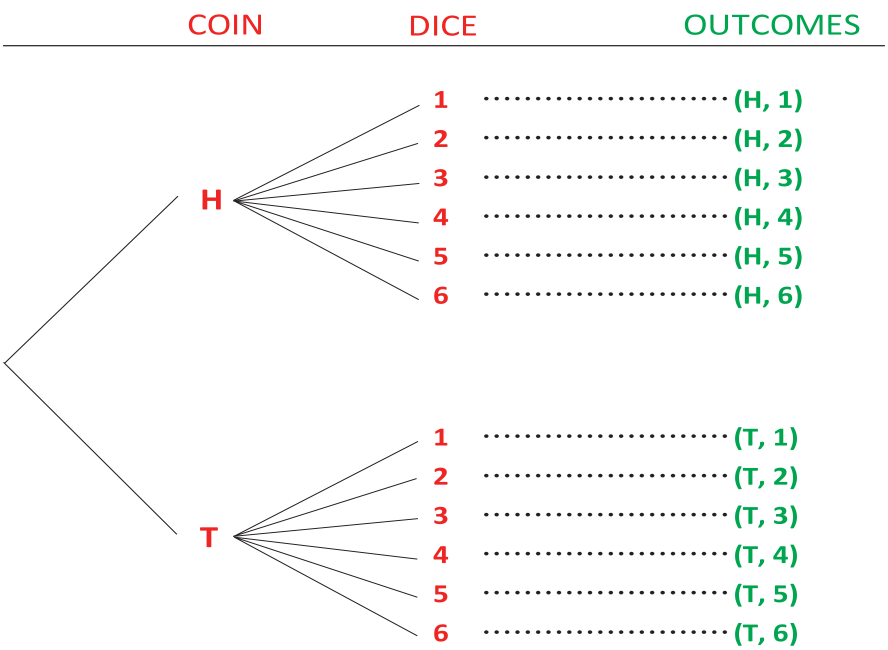


Figure 9.11: A tree diagram in tossing a coin and a die

b Probability of getting a tail and an odd number $= \frac{3}{12} = \frac{1}{4}$

c Probability of getting a head and a number less than or equal to 5 is $= \frac{5}{12}$

Exercise 9.21

1. A coin and die are tossed together.
   a) Find the probability of getting a head and even number.

b) Find the probability of getting a number less than 2

2. A bag contains a red, blue and black ball. You run an experiment where you pick a ball from the bag at random and put it back to the bag after confirming which color is picked. You run this experiment twice.

   a) Find the probability of getting a red ball once

   b) Find the probability of getting two balls of the same color

   c) Find the probability of getting two balls of different color

Exercise 9.22

1. You toss a die once.

   a. What is the sample space?

   b. Write all possible elements if the event is getting less than 4.

2. Find the probability of not getting a tail when you toss two fair coins together.

3. Two dice are simultaneously thrown once. Write all possible elements of the following events.

   a. The sum of the numbers facing up is 8.

   b. The product of the two numbers facing up is 2.

   c. Consecutive numbers facing up

4. Three coins are tossed at the same time. Sketch a tree diagram for the outcomes of this experiment. Write the sample space.

5. There are 30 adults and 20 kids in a room. If you randomly select 1 person from the room, what is the probability of selecting a kid?

6. A bag contains five white balls, four black balls and six red balls. A ball is drawn out of the box at random. What is the probability of drawing the ball for:

   a. Red? b. black? c. white?

7. Two dice are thrown together. What is the probability that the number obtained on one of the dice facing up is a multiple of the number obtained on the other dice facing up?

Summary

1. Statistics is the science of collecting, organizing, presenting and interpreting data in order to have a conclusion.
2. A population is a complete collection of individuals, objects or measurements that have common characteristics.
3. A subset of a population is called a sample.
4. With **descriptive statistics**, your goal is to describe the data that you find in a sample or is given in a problem.
5. **Primary Data** is a data that has been generated by the researcher himself/herself, surveys, interviews, experiments, specially designed for understanding and solving the research problem at hand.
6. **Secondary Data is** a data used after it is generated by large government institutions, healthcare facilities etc. as part of organizational record keeping. The data is then extracted from more varied data files.
7. If a classification is not based on numbers, then it is called a qualitative classification.
8. If a classification is based on numbers, then it is called a quantitative classification.
9. With **inference statistics**, your goal is to use the data in a sample to draw conclusions about a larger population.
10. A statistical table is a systematic presentation of data in columns and rows.
11. A histogram is a graphical representation of a frequency distribution in which the variable (V) is plotted on the x-axis and the frequency (f) is plotted on the y-axis.
12. A frequency distribution is a distribution showing the number of observations associated with each data value.
13. The **mean** is the arithmetic average of the data.

14. If $x_1, x_2, \dots\dots. x_n$ are n observations, then the mean, $\bar{x} = \frac{x_1 + x_2 + \cdots \dots + x_n}{n}$.

15. The **median** is the number in the middle of a dataset. When the data has an odd number of counts, the median is the middle number after the data have been ordered. When the data has an even number of counts, the median is the average of the two most central numbers.

16. The **mode** is the most often occurring value in the data.

17. The outcomes of an experiment are said to be equally likely if, when the experiment is repeated a large number of times, each outcome occurs equally often.

18. The sample space for an experiment is the set of all possible outcomes of the experiment.

19. If S is the sample space of an experiment and each element of E is equally likely to occur, then the probability of the event E occurring, denoted by $P(E)$, is defined as:

$$P(E) = \frac{\text{number of elements in E}}{\text{number of elements in S}} = \frac{\text{n(E)}}{\text{n(S)}}$$

Review Exercise

1. What does presenting and tabulating data mean?
2. The amount of money in birr that 20 people have in their pocket are given below: 4,9,2,3,9,1,2,4,2,4,6,6,7,2,5,9,3,5,8,9
   a. Construct a frequency distribution table
   b. What percent of the people have less than Birr 4?
   c. Draw a histogram to represent the data.
3. A bag contains nine balls distinguishable only by their colors; four are blue and five are red. You draw one ball and record their colors.
   a. What is the sample space?
   b. Express the event that the ball drawn is red as a subset of the sample space.
4. A fair coin is tossed, and a fair die is thrown. Write down sample spaces for
   a. tossing of the coin;
   b. throwing of the die;
   c. combination of these experiments.
   d. Let A be the event that a head is tossed, and B be the event that an odd number is thrown. Calculate the probability of the occurrence of A and B by referring to the sample space. Calculate the probability of the occurrence of A or B by referring to the sample space.
5. Refer to the following information to answer the questions that follow.
   Find the **mean, median, mode, range, variance and standard deviation** for the mean temperatures recorded over a five-day period last winter:
   $18, 22, 19, 25, 12.$
6. Find the **mean**, **median, mode, range, variance and standard deviation** of the scores of an exam out of 20% whose distribution is given in the table below.

v	11	12	13	14	15	16
f	6	7	5	7	3	2

7. Which of the following is true?

A. The mean, mode and median of a data cannot be equal.

B. The range of a data cannot be a non-positive number.

C. The sum of the deviations of each value of a population from the mean will always be zero.

8. A fair die is rolled. What is the probability that a 1, 4, 5, or 6 will be on the upper face?

9. A spinner is divided into 3 equal parts, with parts labeled 8, 9, and 10.

i. What is the probability of spinning an 8 on the spinner if you know the arrow landed on an even number

ii. What is the probability of spinning 9 on the spinner if you know the arrow landed on an odd number

10. A pair of dice is rolled. Find the probability that the product of the numbers on the upper faces is:

i. 1

ii. less than 6

iii. odd

iv. greater than or equal to 15

v. less than 2

11. A card is drawn from a well shuffled pack of 52 cards. What is the probability of getting queen of club or a king of hearts card?

12. The points received from the first 7 games of a football club were respectively 1, 3, 1, 3, 3, 1, and 3. What must be the point in the 8^{th} game to have a mean of 2 points?

A. 0 **B.** 1 **C.** 3 **D.** No answer

13. Refer to the frequency distribution table below to answer question that follows:

v	9	6	5	4	3	2
f	6	7	5	7	3	2

Which of the following is true?

A. the mean is 5.4

B. the mode is 9

C. the median is 5.5

D. No answer

Trigonometric table

Corresponding angles (in degree)

↓	sin	cos	tan		
0	0	1	0		90
1	0.017452	0.999848	0.017455	57.2900	89
2	0.03490	0.999391	0.034921	28.63628	88
3	0.052336	0.99863	0.052408	19.08115	87
4	0.069756	0.997564	0.069927	14.30068	86
5	0.087156	0.996195	0.087489	11.43006	85
6	0.104528	0.994522	0.10510	9.514373	84
7	0.121869	0.992546	0.122784	8.144353	83
8	0.139173	0.990268	0.140541	7.115376	82
9	0.156434	0.987688	0.158384	6.313757	81
10	0.173648	0.984808	0.176327	5.671287	80
11	0.190809	0.981627	0.19438	5.144558	79
12	0.207912	0.978148	0.212556	4.704634	78
13	0.224951	0.97437	0.230868	4.33148	77
14	0.241922	0.97030	0.249328	4.010784	76
15	0.258819	0.965926	0.267949	3.732054	75
16	0.275637	0.961262	0.286745	3.487418	74
17	0.292371	0.95630	0.30573	3.270856	73
18	0.309017	0.951057	0.324919	3.077686	72
19	0.325568	0.945519	0.344327	2.904214	71
20	0.34202	0.939693	0.36397	2.74748	70
21	0.358368	0.933581	0.383864	2.605091	69
22	0.374606	0.927184	0.404026	2.475089	68
23	0.390731	0.920505	0.424474	2.355855	67
24	0.406736	0.913546	0.445228	2.246039	66
25	0.422618	0.906308	0.466307	2.144509	65
26	0.438371	0.898794	0.487732	2.050306	64
27	0.45399	0.891007	0.509525	1.962612	63
28	0.469471	0.882948	0.531709	1.880728	62
29	0.484809	0.87462	0.554308	1.80405	61
30	0.5000	0.866026	0.57735	1.732053	60
31	0.515038	0.857168	0.60086	1.664281	59
32	0.529919	0.848048	0.624869	1.600336	58
33	0.544639	0.838671	0.649407	1.539867	57
34	0.559192	0.829038	0.674508	1.482563	56
35	0.573576	0.819152	0.700207	1.42815	55
36	0.587785	0.809017	0.726542	1.376383	54
37	0.601815	0.798636	0.753553	1.327046	53
38	0.615661	0.788011	0.781285	1.279943	52
39	0.62932	0.777146	0.809783	1.23490	51
40	0.642787	0.766045	0.83910	1.191755	50
41	0.656059	0.75471	0.869286	1.15037	49
42	0.66913	0.743145	0.90040	1.110614	48
43	0.68200	0.731354	0.932514	1.07237	47
44	0.694658	0.71934	0.965688	1.035532	46
45	0.707106	0.707107	1.00000	1.00000	45
	cos	sin		tan	↑